THE INTRINSIC DERIVATIVE AND CENTRIFUGAL FORCES IN GENERAL RELATIVITY: I. THEORETICAL FOUNDATIONS

DONATO BINI

Istituto per Applicazioni della Matematica C.N.R., I–80131 Napoli, Italy and International Center for Relativistic Astrophysics, University of Rome, I–00185 Roma, Italy

PAOLO CARINI*

GP-B, Hansen Labs, Stanford University, Stanford, CA 94305, USA and International Center for Relativistic Astrophysics, University of Rome, I–00185 Roma, Italy

ROBERT T. JANTZEN

Department of Mathematical Sciences, Villanova University, Villanova, PA 19085, USA and International Center for Relativistic Astrophysics, University of Rome, I–00185 Roma, Italy

Received 17 December 1996

Everyday experience with centrifugal forces has always guided thinking on the close relationship between gravitational forces and accelerated systems of reference. Once spatial gravitational forces and accelerations are introduced into general relativity through a splitting of spacetime into space-plus-time associated with a family of test observers, one may further split the local rest space of those observers with respect to the direction of relative motion of a test particle world line in order to define longitudinal and transverse accelerations as well. The intrinsic covariant derivative (induced connection) along such a world line is the appropriate mathematical tool to analyze this problem, and by modifying this operator to correspond to the observer measurements, one understands more clearly the work of Abramowicz et al who define an "optical centrifugal force" in static axisymmetric spacetimes and attempt to generalize it and other inertial forces to arbitrary spacetimes. In a companion article the application of this framework to some familiar stationary axisymmetric spacetimes helps give a more intuitive picture of their rotational features including spin precession effects, and puts related work of de Felice and others on circular orbits in black hole spacetimes into a more general context.

Keywords: gravitoelectromagnetism–inertial forces

1. Introduction

Ever since Einstein unified space and time into spacetime, people have been trying to break them apart again. Spacetime splittings play an important role in many aspects of gravitational theory, not only in helping interpret 4-dimensional geometry in terms of our more familiar space-plus-time perspective, but also in mathematical analysis of various problems of the theory. Although there are many variations on the idea of reintroducing space and time, all share a common foundation of introducing a family of test observers in spacetime who measure spacetime

*Present Address: Physics Department, Amherst College, Amherst, MA 01002, USA.

quantities mathematically by the orthogonal decomposition of the tangent spaces to the spacetime manifold into their local rest spaces and local time directions.[1,2] Such a construction leads to a "reference frame" or "reference system" or other variations of this terminology whose effect in general is to contribute inertial forces to the spatial force equation due to the motion of the family of test observers. Although one can agree on these concepts in simple nonrelativistic situations, the richness of general relativity and of the geometry of spacetime allows many variations of their possible generalizations to the latter theory. There is not necessarily a single "correct" generalization of any given concept, but simply different ways of measuring different quantities, some of which may be more useful than others.

Because of the equivalence principle, inertial forces have always entrigued people in connection with gravitational theory. Indeed the concept of centrifugal force is useful in nonrelativistic mechanics, and in recent years Abramowicz et al[3–9] have shown that a certain generalization of this concept can give a nice physical interpretation to certain properties of strong static gravitational fields in general relativity, although attempts to extend it first to stationary and then to arbitrary spacetimes have been somewhat problematic.[10–17] Here we place that work and related studies of de Felice and others[18–24] in the more general context of gravitoelectromagnetism, the framework which encompasses all the various splitting approaches to general relativity and provides a clean description of the possible choices of curved spacetime generalizations of centripetal acceleration and centrifugal and Coriolis forces. This is done not to exaggerate the importance of splitting spacetime but to help clarify the link between our three-dimensional world view and nonrelativistic common experience on the one hand and the interpretation of concepts related to rotation and acceleration in general relativity on the other.

2. The nonrelativistic background

Before exploring the spacetime picture, it is worth recalling the classic example from nonrelativistic mechanics of a rigidly rotating Cartesian coordinate system and its relation to the ideas of centrifugal force and centripetal acceleration. It is here that all our intuition for these concepts has its roots.

Let $x^i = R^i{}_j(t)\bar{x}^j$ be the coordinate transformation between "space-fixed" orthonormal coordinates $\{\bar{x}^i\}$ and rotating "body-fixed" such coordinates $\{x^i\}$ with a common origin in Euclidean space, borrowing from the usual terminology of the rigid body problem in classical mechanics. The body-fixed components of the angular velocity of the rotating system are defined by

$$\dot{R}^i{}_k R^{-1\,k}{}_j = \delta^{ik}\epsilon_{kjm}\Omega^m \ . \tag{2.1}$$

Both these and the space-fixed components $\bar{\Omega}^i = R^{-1\,i}{}_j\Omega^j$ are constant in the case of a rotation about a fixed axis with constant angular velocity about that axis.

If one evaluates the first and second time derivatives of the coordinates of a trajectory, one easily finds

$$\dot{\bar{x}}^i = R^{-1\,i}{}_j[V^j + \dot{x}^j] \ ,$$

$$\ddot{\bar{x}}^i \quad = \quad {R^{-1}}^i{}_j[A^j + 2(\Omega \times \dot{x})^j + \ddot{x}^j] \ , \tag{2.2}$$

where $V^i = (\Omega \times x)^i = \delta^{ij}\epsilon_{jkm}\Omega^k x^m$ is the relative velocity field of the body-fixed points relative to the space-fixed points, and $A^i = (\Omega \times (\Omega \times x))^i + (\dot{\Omega} \times x)^i$ is the relative acceleration field, both evaluated along the trajectory.

If $\ddot{\bar{x}}^i = \bar{F}^i$ is the space-fixed force per unit mass equation of motion for a point particle of mass m moving under the influence of a force $m\bar{F}^i$, then the acceleration relation above may be rewritten in the form

$$\ddot{x}^i = F^i + g^i + (\dot{x} \times H)^i \ , \tag{2.3}$$

by moving the additional terms due to the time dependence of the transformation to the other side of the equation where they appear as "inertial forces" due to the motion of the body-fixed points to which the coordinate system is attached. One has a "gravitoelectric-like" force $g^i = -A^i$ due to the acceleration of those points, and the remaining "gravitomagnetic-like" force, the Coriolis force, which is due to the changing orientation of the body-fixed axes, which in turn is a manifestation of the relative motion of the body-fixed points. The latter force involves a "gravitomagnetic-like field" $H^i = 2\Omega^i$. The "gravitoelectric-like" force consists of the centrifugal force directed away from the axis of rotation and an additional force due to the changing angular velocity.

In the more familiar case of a time-independent angular velocity, these two noninertial force fields admit a scalar and a vector potential respectively. The scalar potential is just half the square of the magnitude of the relative velocity field

$$\begin{aligned} \Phi &= \frac{1}{2}\delta_{ij}V^iV^j \ , \\ g^i &= -\delta^{ij}\,\mathrm{grad}_j\,\Phi \ , \end{aligned} \tag{2.4}$$

while this field itself serves as the vector potential

$$H^i = (\mathrm{curl}\,V)^i \ . \tag{2.5}$$

Since each body-fixed point is undergoing circular motion at constant velocity, the relative acceleration field is exactly the familiar centripetal acceleration associated with this simple motion, so the centrifugal force is just the sign-reversal of this centripetal acceleration. The Coriolis force can be interpreted as due to the local vorticity of the flow of the body-fixed points in space (half the curl of the velocity field), which is equal to the global constant angular velocity vector Ω^i.

3. The Spacetime Setting

The great simplication of this discussion compared to a corresponding one in terms of a geometric splitting of spacetime is the common Newtonian time used by both systems of spatial coordinates. In a spacetime discussion, one must also take into account the change in the local time direction, which complicates matters, especially the relationship between a sign-reversed centripetal acceleration and a corresponding centrifugal force. One also must re-interpret the time derivative in a way which makes geometric sense, and there are a number of distinct ways of doing this, depending on how changes in fields are measured along general world lines.

Consider only the case of time-independent angular velocity. By adding the additional time coordinate transformation

$$t = \bar{t} \tag{3.1}$$

to the original spatial coordinate transformation, one obtains a transformation from inertial coordinates $\{\bar{t}, \bar{x}^i\}$ in Minkowski spacetime to noninertial coordinates $\{t, x^i\}$ which may be interpreted using the spacetime geometry. The time lines of each such coordinate system sharing the same time coordinate hypersurfaces in Minkowski spacetime may be interpreted as the world lines of a family of test observers (when timelike), representing the trajectories of the space-fixed and body-fixed points. The first set are inertial (zero acceleration) observers with zero relative velocity, whose world lines are the orthogonal trajectories to the family of time hypersurfaces, while the second are noninertial (accelerated) observers in relative motion and not admitting any orthogonal family of hypersurfaces. Both families of world lines are the flow lines of Killing vector fields of Minkowski spacetime (therefore having zero expansion tensor), the first with zero vorticity and the second with nonzero vorticity.

In this description, it is only the families of time lines and time hypersurfaces which play a central role, not the specific choice of spatial coordinates which parametrize the families of time lines. It is in fact useful to introduce new spatial coordinates adapted to the orbits of the body-fixed points, and their corresponding rotating system, namely nonrotating and rotating cylindrical coordinates adapted to the axis of the rotation. For example, if one chooses the axis of rotation so that $\bar{\Omega}^i = \Omega^i = \Omega \delta^i{}_3$, one can choose the usual cylindrical coordinates $\{\bar{\rho}, \bar{\phi}, \bar{z}\}$ in place of $\{\bar{x}^i\}$, and rotating cylindrical coordinates $\{\rho, \phi, z\}$ which differ only by $\phi = \bar{\phi} - \Omega \bar{t}$.

However, a description in terms of quantities measured by each family of test observers involves their local proper times, which in the second case are not associated with any global time function. Global time functions not directly measuring observer proper time occur in the context of "observer-adapted" coordinate systems. A system of coordinates can be adapted to the observer congruence in one of two ways.[1] For a general congruence with nonzero vorticity, the appropriate coordinates are comoving, leading to an approach called the threading point of view, while for a vorticity-free congruence, one can use more general coordinates adapted to the family of orthogonal hypersurfaces admitted by the congruence, leading to an approach called the slicing point of view. In each case, the adapted local coordinates $\{t, x^i\}$ lead to the introduction of explicit potentials for the various gravitational forces.

A slicing together with a transversal threading (congruence) describes exactly the structure on which each of these two points of view is built, here called a "nonlinear reference frame." In the flat Minkowski space example with inertial or rotating observers, one has the geodesically parallel slicing orthogonal to the paths of the original inertial observers, together with the rotating observers which provide a new threading of this slicing (when those observers are defined). Rotating Cartesian or cylindrical coordinates on Minkowski spacetime are examples of coordinates adapted to the nonlinear reference frame associated with this slicing and threading. In the threading point of view the threading congruence serves both as the observer congruence as well as the curves along which evolution is measured, while in the slicing point of view these two roles are separated: the orthogonal trajectories to the slicing form the observer congruence, while the measured quantities are evolved along the distinct threading congruence.

This is a nonlinear reference frame adapted to the stationary axisymmetry of the flat Minkowski spacetime. A similar geometrically privileged nonlinear reference frame exists in any stationary axisymmetric spacetime, in particular in the Kerr black hole spacetimes (with the Schwarzschild black hole as their static limit) and in the Gödel spacetime. For such spacetimes with some of the same symmetries which characterize the uniformly rotating observers in Minkowski spacetime, one might hope to define generalized centrifugal and Coriolis forces, but without these symmetries one must rethink the point of departure.

This is not the whole story about centrifugal force in classical mechanics, since it makes its appearance in at least two other familiar contexts. As discussed in detail by Abramowicz,[5] one is the train, plane, car context in which one has an accelerated platform to which a local reference frame is attached, essentially the previous problem with additional motion of the origin of coordinates. Any point fixed in this local platform will then experience accelerations tangential and transverse to its direction of motion, and the transverse acceleration can be interpreted in terms of a centrifugal force in the local reference frame due to its instantaneous rotation about the center of the osculating circle associated with the curvature of its path. If the point is in motion with respect to the local platform, Coriolis effects are felt as well, but the details are more complicated than the simpler rigid body discussion.

A third context in which the centrifugal force is usually introduced is in the discussion of motion in a central potential.[5] Here one introduces a polar coordinate system in the plane of the motion and then expresses the equation of motion in that coordinate system. The radial component of this equation for general motion then contains what is interpreted as a centrifugal force term due to the curvature of the circular angular coordinate lines. This term is just the sign-reversal of the centripetal acceleration for motion confined to these coordinate lines and enters the equation of motion through a Christoffel symbol term associated with this curvature, quadratic in the angular speed. Although there is no rotating frame in this discussion, one may be introduced by letting the new system rotate about the center of force so that the particle in motion has a fixed new angular coordinate. In this rotating frame, the same centrifugal force term is then realized as in the rigid body discussion as a rotating frame effect, but rotating about the center of force, not the instantaneous radial direction associated with the curvature of the particle path. For circular motion, this centrifugal force is just the sign-reversal of the centripetal acceleration of the particle path, but for general noncircular motion, the two quantities are not simply linked.

In other words, the "fictitious" centrifugal force is a convenience that only has meaning with respect to some implied reference frame, and in the same problem can play different roles depending on which frame is chosen. For circular trajectories, all of these various aspects of centrifugal force come together, presenting the most useful application of the concept.

In general relativity the best hope of having a useful generalization of centrifugal force lies in the static axisymmetric case with circular trajectories. Consideration of noncircular trajectories in that case or relaxing the symmetry even to stationarity already introduces difficulties which make the whole discussion rather unclear. However, the idea of a relative centripetal acceleration viewed by an observer, being well defined in a single reference frame, sidesteps the questions involving two different reference frames that seem to be tied up with various aspects of centrifugal force and it is therefore reasonable to generalize it to an arbitrary spacetime.

The key difference between the nonrelativistic description of inertial forces and the framework of general relativity is that in the latter context, the effects of a gravitational field due to the presence of matter are intertwined with those of the accelerated motion of the field of observers used to establish a given reference frame in order to "measure" the gravitational forces. Only under special symmetry conditions does it seem to make sense to try to separate the two. Moreover, focusing attention on a single test particle world line without referring it to an independent family of test observers (which are not adapted to the particular worldline) reduces the usefulness of a space-plus-time interpretation of the motion, except possibly in reference to the spacetime Frenet-Serret frame which is completely determined by the world line alone.[25–28]

4. A family of test observers

A splitting of a general spacetime equipped with a Lorentzian metric $g_{\alpha\beta}$ (signature -+++) and the covariant derivative ∇_α associated with its symmetric connection is accomplished locally by specifying a future-pointing unit timelike vector field u^α ($u^\alpha u_\alpha = -1$). This field may be interpreted as the four-velocity of a family of test observers whose proper time parametrized world lines (let τ_u denote such a parameter on each world line) are integral curves of u^α.

The orthogonal decomposition of each tangent space into the local rest space and local time direction of the observer extends to all the tensor spaces above it and to the algebra of spacetime tensor fields, and may be referred to as the measurement process associated with the family of test observers. Tensors or tensor fields which have no component along u^α are called spatial (with respect to u^α). The fully covariant and contravariant forms of the spatial projection tensor $P(u)^\alpha{}_\beta =$

$\delta^\alpha{}_\beta + u^\alpha u_\beta$ are referred to as the corresponding forms of the spatial metric

$$P(u)_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta \ . \tag{4.1}$$

Similarly let $\eta(u)_{\alpha\beta\delta} = u^\delta \eta_{\delta\alpha\beta\delta}$ be the spatial unit antisymmetric tensor associated with the spatial metric, where $\eta_{0123} = 1 = \eta(u)_{123}$ in a time-oriented, oriented orthonormal frame having u^α as its first element. This tensor may be used to define a spatial duality operation for antisymmetric spatial tensor fields in an obvious way.

As described in Ref. 1, one may also spatially project various derivative operators so that the result of the derivative of any tensor field is always spatial; such derivatives naturally occur in expressing tensor equations in space-plus-time form. Two useful spatial derivatives are the spatial Lie derivative $\pounds(u)_X = P(u)\pounds_X$, for any spatial vector field X^α, and the spatial covariant derivative $\nabla(u)_\alpha = P(u)P(u)^\beta{}_\alpha \nabla_\beta$, where the projection on all free indices is implied after the application of the derivative. Similarly three useful temporal derivatives are the temporal Lie derivative $\nabla_{(\mathrm{lie})}(u) = P(u)\pounds_u$, the Fermi-Walker temporal derivative $\nabla_{(\mathrm{fw})}(u) = P(u)u^\alpha \nabla_\alpha$, and the corotating Fermi-Walker temporal derivative $\nabla_{(\mathrm{cfw})}(u)$ related to the first two by the kinematical fields (acceleration $a(u)^\alpha$, vorticity $\omega(u)^\alpha{}_\beta$, expansion $\theta(u)^\alpha{}_\beta$) of the observer congruence,

$$\begin{aligned} a(u)^\alpha &= \nabla_{(\mathrm{fw})}(u) u^\alpha \ , \\ \omega(u)_{\alpha\beta} &= P(u)^\gamma{}_\alpha P(u)^\delta{}_\beta \nabla(u)_{[\gamma} u_{\delta]} \ , \\ \theta(u)_{\alpha\beta} &= P(u)^\gamma{}_\alpha P(u)^\delta{}_\beta \nabla(u)_{(\gamma} u_{\delta)} \\ &= \frac{1}{2}\nabla_{(\mathrm{lie})}(u) g_{\alpha\beta} = \frac{1}{2}\nabla_{(\mathrm{lie})}(u) P(u)_{\alpha\beta} \ , \end{aligned} \tag{4.2}$$

through the relations

$$\nabla_{(\mathrm{cfw})}(u) X^\alpha = \nabla_{(\mathrm{fw})}(u) X^\alpha + \omega(u)^\alpha{}_\beta X^\beta = \nabla_{(\mathrm{lie})}(u) X^\alpha + \theta(u)^\alpha{}_\beta X^\beta \tag{4.3}$$

valid only for a spatial vector field X^α, but easily extended to any spatial tensor field in the usual way. It is also convenient to use extend the notation $\pounds(u)_X = P(u)\pounds_X$ to any vector field X^α, spatial or not. All indices are spatially projected when these spatial differential operators are applied to a tensor field. The geometrical meaning of these derivatives is discussed in Ref. 1. For spatial fields, the Fermi-Walker temporal derivative coincides with the spacetime Fermi-Walker derivative along the observer congruence, explaining the terminology.

Note that the spatial covariant derivative and the ordinary and corotating Fermi-Walker temporal derivatives of the spatial metric are all zero, so index-shifting of spatial fields commutes with these derivatives. The Lie temporal derivative of the spatial metric instead equals twice the expansion tensor of the observer congruence, so index shifting of a spatial tensor being differentiated by this operator leads to additional terms involving this tensor.

It is useful to introduce a vorticity or rotation vector field using the spatial duality operation

$$\omega(u)^\alpha = \tfrac{1}{2}\eta(u)^{\alpha\beta\gamma}\omega(u)_{\beta\gamma} \ . \tag{4.4}$$

This in turn may be used to rewrite the contraction of the vorticity tensor with a spatial vector field as a spatial cross-product

$$\omega(u)^\alpha{}_\beta X^\beta = -\eta(u)^\alpha{}_{\beta\gamma}\omega(u)^\beta X^\gamma = -[\vec{\omega}(u) \times_u X]^\alpha \ . \tag{4.5}$$

Finally the shear tensor is defined as the spatial tracefree part of the expansion tensor

$$\sigma(u)_{\alpha\beta} = \theta(u)_{\alpha\beta} - \tfrac{1}{3}\theta(u)^\gamma{}_\gamma P(u)_{\alpha\beta} \ . \tag{4.6}$$

5. Measuring the intrinsic derivative along a parametrized curve

Given any parametrized curve in spacetime, with parameter λ and tangent $V(\lambda)^\alpha$, the spacetime connection induces a connection on the curve whose derivative is called either the "intrinsic" or "absolute" derivative along it.[29] This derivative $D/d\lambda$ is uniquely defined by the condition that if one extends a tensor smoothly off the curve to a tensor field on spacetime, the action of this intrinsic derivative on the tensor at a point on the curve equals the action of the covariant directional derivative $V(\lambda)^\alpha\nabla_\alpha$ on the extended tensor field at that point

$$DT^{\alpha\dots}{}_{\beta\dots}/d\lambda = V(\lambda)^\gamma\nabla_\gamma T^{\alpha\dots}{}_{\beta\dots} \ . \tag{5.1}$$

In this equation it is important to note that its right hand side is understood to be the value on the worldline of the derivative of the extended tensor field. The usual sloppy notation of this equation which does not distinguish between the original and the extended tensor field nor indicate its validity only on the curve itself must always be understood in this context.

For a vector defined only along the curve, this leads to the usual formula in terms of the ordinary parameter derivative of the components along the world line plus the connection coefficient correctional terms

$$DX^\alpha/d\lambda = dX^\alpha/d\lambda + \Gamma^\alpha{}_{\beta\gamma}V(\lambda)^\beta X^\gamma \ . \tag{5.2}$$

In a previous article[1] this derivative has been referred to as the total covariant derivative along $V(\lambda)^\alpha$ or along the parametrized curve.

If one performs an orthogonal projection of the intrinsic derivative along such a parametrized curve in the measurement process associated with the family of test observers, one is led to a new derivative operator along that curve which is natural to call either the Fermi-Walker spatial intrinsic derivative or the Fermi-Walker total spatial covariant derivative. For example, if X^α is any vector defined along the curve, this derivative is defined by

$$D_{(\mathrm{fw})}(V(\lambda),u)X^\alpha/d\lambda = P(u)^\alpha{}_\beta DX^\beta/d\lambda \ . \tag{5.3}$$

If one extends X^α to a smooth vector field off the curve, then for this extended vector field, expressing this derivative in terms of the measurement of the tangent vector itself

$$V(\lambda)^\alpha = V(\lambda)^{(||_u)}\, u^\alpha + [P(u)V(\lambda)]^\alpha \tag{5.4}$$

leads to the natural pairing of its temporal part $V(\lambda)^{(||_u)} = -u_\gamma V(\lambda)^\gamma$ and spatial part $[P(u)V(\lambda)]^\alpha = P(u)^\alpha{}_\gamma V(\lambda)^\gamma$ with the corresponding projections of the covariant derivative acting on the smooth extension

$$D_{(\mathrm{fw})}(V(\lambda), u)X^\alpha/d\lambda = \{V(\lambda)^{(||_u)}\nabla_{(\mathrm{fw})}(u) + [P(u)V(\lambda)]^\beta \nabla(u)_\beta\}X^\alpha \ , \tag{5.5}$$

where the individual terms on the right hand side depend on how the extension is made. These individual terms have no meaning without such an extension, a fact which has caused some confusion in attempts to generalize centrifugal and other noninertial forces to curved spacetimes.[12–15]

By replacing the Fermi-Walker temporal derivative in the expression for the Fermi-Walker intrinsic derivative acting on the extended tensor field with the Lie temporal derivative or the corotating Fermi-Walker temporal derivative respectively, one defines the corresponding total spatial covariant derivatives of a spatial vector field

$$\begin{aligned} D_{(\mathrm{tem})}(V(\lambda), u)X^\alpha/d\lambda &= \{V(\lambda)^{(||_u)}\nabla_{(\mathrm{tem})}(u) + [P(u)V(\lambda)]^\beta \nabla(u)_\beta\}X^\alpha \\ &\qquad \mathrm{tem}=\mathrm{fw}, \mathrm{cfw}, \mathrm{lie}\ , \end{aligned} \tag{5.6}$$

where again the righthand side expression only makes sense for an extended vector field, but leads to well-defined derivatives for a spatial vector defined only on the parametrized curve.

These three derivatives of spatial vectors along the world line differ among themselves only by a linear transformation of the local rest space

$$\begin{aligned} D_{(\mathrm{cfw})}(V(\lambda), u)X^\alpha/d\lambda &= D_{(\mathrm{fw})}(V(\lambda), u)X^\alpha/d\lambda + V(\lambda)^{(||_u)}\omega(u)^\alpha{}_\beta X^\alpha \\ &= D_{(\mathrm{lie})}(V(\lambda), u)X^\alpha/d\lambda + V(\lambda)^{(||_u)}\theta(u)^\alpha{}_\beta X^\alpha \ , \end{aligned} \tag{5.7}$$

expressions which may be used together with Eq. (5.3) to define the corotating and Lie such derivatives in terms of the Fermi-Walker one when acting on spatial fields. The ordinary and corotating Fermi-Walker total spatial covariant derivatives of the spatial metric vanish so they commute with index shifting on spatial fields, and one may introduce spatial orthonormal triads along the curve for which one of these derivatives vanishes. These are natural to call "relative Fermi-Walker" or "relative corotating Fermi-Walker" propagated spatial frames along the parametrized curve. The Lie such derivative of the spatial metric vanishes only if the expansion tensor vanishes, in which case it coincides with the corotating Fermi-Walker total spatial covariant derivative. Thus in general the "relative Lie" propagated spatial frames along the parametrized curve will not remain orthonormal if initially so when the expansion tensor is nonzero.

The three different choices of derivative correspond to the three possible ways of evolving spatial frames into the future, the first two of which preserve inner products. They each measure differences with respect to the associated relative propagated spatial frames along the given curve. The relationship between the relative Lie transport and the relative Fermi-Walker transport along the observer congruence itself leads to the physical intepretation of the rotation, expansion, and shear of that congruence.

6. Reparametrization of a parametrized curve

Two new parametrizations may be introduced for any parametrized curve by solving the following ordinary differential equations

$$d\tau_{(V(\lambda),u)}/d\lambda = V(\lambda)^{(||_u)} \ , \quad d\ell_{(V(\lambda),u)}/d\lambda = ||P(u)V(\lambda)|| \ , \tag{6.1}$$

(using the notation $||Y|| = |Y_\alpha Y^\alpha|^{1/2}$) corresponding to the limiting sequence of temporal and spatial arclength differentials seen by the test observers whose paths are crossed by the curve. The solutions lead to valid reparametrizations as long as the right hand side of the differential equation does not vanish, so that an invertible relationship exists between the old and new parametrizations. When one of the two right hand sides vanishes identically, the parameter becomes a proper interval parameter (proper distance orthogonal to the observer family or proper time along it respectively).

The derivatives of these new parameters are in turn related to the derivatives of the spacetime interval by the usual relation

$$[ds/d\lambda]^2 = -[d\tau_{(V(\lambda),u)}/d\lambda]^2 + [d\ell_{(V(\lambda),u)}/d\lambda]^2 \ , \tag{6.2}$$

while their quotient up to sign defines the relative speed of the curve as seen by the observer family

$$\pm\nu(V(\lambda),u) = d\ell_{(V(\lambda),u)}/d\tau_{(V(\lambda),u)} = ||P(u)V(\lambda)||/V(\lambda)^{(||_u)} \ . \tag{6.3}$$

The relative velocity itself and (when nonzero) the unit vector defining its direction are themselves defined by

$$\nu(V(\lambda),u)^\alpha = [P(u)V(\lambda)]^\alpha / V(\lambda)^{(||_u)} \ , \tag{6.4}$$

and

$$\hat{\nu}(V(\lambda),u)^\alpha = [P(u)V(\lambda)]^\alpha / ||P(u)V(\lambda)|| \ . \tag{6.5}$$

The total spatial covariant derivatives along the parametrized curve may be re-expressed in terms of the new parametrizations by the chain rule

$$D/d\lambda' = [d\lambda/d\lambda']\, D/d\lambda = [d\lambda'/d\lambda]^{-1}\, D/d\lambda \ , \tag{6.6}$$

where D here stands for any of the intrinsic derivative operators and λ' for either new parametrization.

Since the Fermi-Walker and corotating Fermi-Walker such derivatives respect spatial orthogonality, one may use them to introduce a relative spatial Frenet-Serret frame of each type along the parametrized curve, using the relative spatial arclength parametrization to generalize the usual objects on a Riemannian 3-manifold. The spatial unit vector $\hat{\nu}(V(\lambda), u)^\alpha$ plays the role of the "relative unit tangent," and the direction and length of its derivative with respect to the relative spatial arclength yields the "relative unit normal" and "relative spatial curvature" of each type, while the spatial cross-product of the relative unit tangent and normal defines the "relative unit bi-normal," whose derivative in turn leads to the "relative torsion." For a stationary spacetime with test observers following the trajectories of timelike Killing vector field, the relative Frenet-Serret structure for the corotating case corresponds to the usual such structure on the observer quotient space with the projected Riemannian metric, suggesting that it is a generalization worth considering. Such a relative spatial Frenet-Serret structure should be clearly distinguished from the spacetime Frenet-Serret structure for the curve.[25–28]

7. Measuring the intrinsic derivative along a test particle world line

Suppose one considers the world line of a nonzero rest mass test particle in spacetime parametrized by the particle's proper time τ_U, letting U^α denote its unit timelike tangent vector, the four-velocity of the test particle. This vector may be split using the orthogonal decomposition associated with the family of test observers with four-velocity u^α

$$U^\alpha = \gamma(U,u)[u^\alpha + \nu(U,u)^\alpha] = E(U,u)u^\alpha + p(U,u)^\alpha \ . \tag{7.1}$$

Here the spatial vector $\nu(U,u)^\alpha$ is the relative velocity of U^α with respect to u^α, and $\gamma(U,u) = [1-\nu(U,u)^2]^{-1/2}$ is its associated gamma factor, while $\nu(U,u) = [\nu(U,u)_\alpha \nu(U,u)^\alpha]^{1/2}$ is the relative speed. Similarly $p(U,u)^\alpha = \gamma(U,u)\nu(U,u)^\alpha$ is the three-momentum (per unit mass) observed by the test observers, with magnitude $p(U,u)$, while $E(U,u) = \gamma(U,u)$ is the energy (per unit mass) as seen by the test observers. (The tilde notation of Ref. 1 used for per unit mass quantities will be dropped for simplicity.) Either set of quantities satisfies the identity

$$\gamma(U,u)^2 = \gamma(U,u)^2\nu(U,u)^2 + 1 \ , \quad E(U,u)^2 = p(U,u)^2 + 1 \ , \tag{7.2}$$

imposed by the unit nature of u^α. Finally the two new relative parametrizations of the world line are here defined by

$$d\tau_{(U,u)}/d\tau_U = \gamma(U,u) \ , \quad d\ell_{(U,u)}/d\tau_U = \gamma(U,u)\nu(U,u) \ , \tag{7.3}$$

with

$$d\ell_{(U,u)}/d\tau_{(U,u)} = \nu(U,u) \ . \tag{7.4}$$

Adopting the observer proper time parametrization, the spatial projection of the total covariant derivative along the world line $D/d\tau_{(U,u)} = \gamma(U,u)^{-1}D/d\tau_U$ defines the Fermi-Walker total spatial covariant derivative, which together with its two spatial generalizations can be expressed in the following way for a spatial vector field X^α defined along the world line and which has been extended off the world line for the right hand side to make sense

$$D_{(\mathrm{tem})}(U,u)X^\alpha/d\tau_{(U,u)} = [\nabla_{(\mathrm{tem})}(u) + \nu(U,u)^\beta \nabla(u)_\beta]X^\alpha \ , \quad \mathrm{tem{=}fw,cfw,lie} \ . \tag{7.5}$$

These three derivatives of spatial vectors along the world line differ among themselves only by a linear transformation of the local rest space

$$
\begin{aligned}
D_{(\mathrm{cfw})}(U,u)X^\alpha/d\tau_{(U,u)} &= D_{(\mathrm{fw})}(U,u)X^\alpha/d\tau_{(U,u)} + \omega(u)^\alpha{}_\beta X^\alpha \\
&= D_{(\mathrm{lie})}(U,u)X^\alpha/d\tau_{(U,u)} + \theta(u)^\alpha{}_\beta X^\alpha \ , \qquad (7.6)
\end{aligned}
$$

expressions which may be used to define the Lie and corotating Fermi-Walker such derivatives in terms of the Fermi-Walker one when acting on spatial fields.

8. Relative acceleration: longitudinal and transverse parts

Applying these derivatives to the relative velocity itself leads to a relative acceleration vector for each one

$$
a_{(\mathrm{tem})}(U,u)^\alpha = D_{(\mathrm{tem})}\nu(U,u)^\alpha/d\tau_{(U,u)} \ , \qquad (8.1)
$$

differing from each other in the same way as the above three derivatives

$$
a_{(\mathrm{cfw})}(U,u)^\alpha = a_{(\mathrm{fw})}(U,u)^\alpha + [\vec\omega(u)\times_u \nu(U,u)]^\alpha = a_{(\mathrm{lie})}(U,u)^\alpha + \theta(u)^\alpha{}_\beta\nu(U,u)^\beta \ . \qquad (8.2)
$$

The first of these equations just reflects the relative rotation of the relative Fermi-Walker and corotating Fermi-Walker transported axes along the world line. Apart from a gamma factor, the rate of change of the spatial momentum is related to the relative acceleration by an additional term along the direction of relative motion involving the rate of change of the energy (per unit mass) $E(U,u)=\gamma(U,u)$ of the test particle

$$
D_{(\mathrm{tem})}(U,u)p(U,u)^\alpha/d\tau_{(U,u)} = \nu(U,u)^\alpha d\ln\gamma(U,u)/d\tau_{(U,u)} + \gamma(U,u)a_{(\mathrm{tem})}(U,u)^\alpha \ . \qquad (8.3)
$$

One may further decompose both the relative acceleration and the observed rate of change of spatial momentum into longitudinal and transverse components with respect to the observed motion of the test particle using the relative motion projectors[1,30]

$$
P_u(U,u)^{(||)\alpha}{}_\beta = \hat\nu(U,u)^\alpha\hat\nu(U,u)_\beta \ , \quad P_u(U,u)^{(\perp)\alpha}{}_\beta = P(u)^\alpha{}_\beta - P_u(U,u)^{(||)\alpha}{}_\beta \ . \qquad (8.4)
$$

For the ordinary and corotating Fermi-Walker cases, this decomposition is equivalent to the terms arising from the product rule when these quantities are represented as the scalar product of their magnitude and direction

$$
\nu(U,u)^\alpha = \nu(U,u)\hat\nu(U,u)^\alpha \ , \quad p(U,u)^\alpha = p(U,u)\hat\nu(U,u)^\alpha \ , \qquad (8.5)
$$

where $p(U,u) = \gamma(U,u)\nu(U,u)$. Consider first the relative accelerations, which decompose into two terms

$$
\begin{aligned}
a_{(\mathrm{tem})}(U,u)^\alpha &= \hat\nu(U,u)^\alpha d\nu(U,u)/d\tau_{(U,u)} + \nu(U,u)D_{(\mathrm{tem})}(U,u)\hat\nu(U,u)^\alpha/d\tau_{(U,u)} \\
&= a^{(||)}_{(\mathrm{tem})}(U,u)^\alpha + a^{(\perp)}_{(\mathrm{tem})}(U,u)^\alpha \ , \quad \mathrm{tem=fw,cfw} \ , \qquad (8.6)
\end{aligned}
$$

which define respectively their components parallel ("tangential" to the observed orbit, or longitudinal) and perpendicular ("normal" or transverse) to the relative direction of motion, most naturally called the longitudinal and transverse relative accelerations, conforming to traditional names for these quantities.

For the Lie case the derivative of the unit relative velocity is not orthogonal to the velocity vector

$$\hat{\nu}(U,u)_\alpha D_{(\rm lie)}(U,u)\hat{\nu}(U,u)^\alpha/d\ell_{(U,u)} = -2\theta(u)_{\alpha\beta}\hat{\nu}(U,u)^\alpha\hat{\nu}(U,u)^\beta \tag{8.7}$$

unless the expansion tensor of the observer congruence vanishes (in which case the Lie and corotating Fermi-Walker derivatives of the various types agree) or unless the relative motion is along a direction in which the observer expansion is zero. Thus one must actually project this derivative in order to accomplish the "direction-of-relative-motion" orthogonal decomposition.

The transverse relative acceleration for the ordinary and corotating Fermi-Walker cases

$$\begin{aligned} a^{(\perp)}_{(\rm tem)}(U,u)^\alpha &= \nu(U,u)D_{(\rm tem)}(U,u)\hat{\nu}(U,u)^\alpha/d\tau_{(U,u)} \\ &= \nu(U,u)^2 D_{(\rm tem)}(U,u)\hat{\nu}(U,u)^\alpha/d\ell_{(U,u)}\ , \end{aligned} \tag{8.8}$$

where Eq. (7.4) has been used to re-parametrize the derivative of the unit velocity vector, is exactly what one calls the centripetal acceleration in the case of the usual inertial observers in Minkowski spacetime, so it is natural to call it the "relative centripetal acceleration."

One may next decompose the rate of change of spatial momentum in the same way

$$\begin{aligned} D_{(\rm tem)}(U,u)p(U,u)^\alpha/d\tau_{(U,u)} &= \hat{\nu}^\alpha dp(U,u)/d\tau_{(U,u)} \\ &\quad +\gamma(U,u)\nu(U,u)D_{(\rm tem)}(U,u)\hat{\nu}(U,u)^\alpha/d\tau_{(U,u)} \\ &= \hat{\nu}^\alpha dp(U,u)/d\tau_{(U,u)} + \gamma(U,u)a^{(\perp)}_{(\rm tem)}(U,u)^\alpha\ , \end{aligned} \tag{8.9}$$

where the second equality only holds for the ordinary and corotating Fermi-Walker cases. The second term is proportional to the relative acceleration for those cases. The first term is parallel to the direction of motion and itself contains both the longitudinal relative acceleration scalar $d\nu(U,u)/d\tau_{(U,u)}$ as well as the effect of the changing three-energy (per unit mass) of the test particle when the derivative is expanded by the product rule.

Consider the derivatives of the unit velocity vector for the two Fermi-Walker cases, which are related to each other by

$$\begin{aligned} D_{(\rm cfw)}(U,u)\hat{\nu}(U,u)^\alpha/d\ell_{(U,u)} &= D_{(\rm fw)}(U,u)\hat{\nu}(U,u)^\alpha/d\ell_{(U,u)} \\ &\quad -\nu(U,u)^{-1}[\vec{\omega}(u)\times_u \hat{\nu}(U,u)]^\alpha\ . \end{aligned} \tag{8.10}$$

If each were the uniquely defined intrinsic derivative with respect to the arclength of the unit tangent to a curve in a three-dimensional Riemannian manifold, the unit vector $\eta_{(\mathrm{tem})}(U,u)^\alpha$ specifying its direction (the relative unit normal) would be the first normal to the curve and its magnitude $\kappa_{(\mathrm{tem})}(U,u) \geq 0$ (the relative curvature) would be the curvature of that curve, the reciprocal of which would define a radius of curvature (the relative radius of curvature) $\rho_{(\mathrm{tem})}(U,u) = 1/\kappa_{(\mathrm{tem})}(U,u)$ when the curvature is nonzero. This leads to the representation

$$D_{(\mathrm{tem})}(U,u)\hat{\nu}(U,u)^\alpha/d\ell_{(U,u)} = 1/\rho_{(\mathrm{tem})}(U,u)\,\eta_{(\mathrm{tem})}(U,u)^\alpha \tag{8.11}$$

of the unit velocity derivative and

$$a^{(\perp)}_{(\mathrm{tem})}(U,u) = \nu(U,u)^2/\rho_{(\mathrm{tem})}(U,u) \tag{8.12}$$

for the magnitude of the relative centripetal acceleration, which takes its familiar form in terms of the relative radius of curvature. Eq. (8.2) shows that the two accelerations differ by a term orthogonal to the relative direction of motion. As noted above, in the stationary case these concepts reduce to the analogous quantities in the Riemannian 3-manifold of the observer quotient space. For an arbitrary spacetime these generalizations can be studied to understand the sense in which they generalize the more familiar concepts, but for now they will be taken merely as formal definitions.

For any test particle trajectory, the relative centripetal acceleration is zero when the relative curvature $\kappa_{(\mathrm{tem})}(U,u)$ vanishes, corresponding to the limit of infinite radius of curvature. This enables one to define trajectories for which the curvature vanishes identically as "relatively straight" with respect to the ordinary or corotating Fermi-Walker total spatial covariant derivative. Relative motion for which this is true may be called the case of purely linear relative acceleration (for each type), examined recently in the static case by Rindler and Mishra[31,32] and the present authors.[30] On the other hand, if the longitudinal relative acceleration vanishes, as it does for the case of constant relative speed in the ordinary and corotating cases, one has the case of purely transverse relative acceleration, the case studied extensively for circular orbits by Abramowicz et al in the static case[3–9] and by Abramowicz and coworkers in the stationary Kerr spacetime and stationary axisymmetric spacetimes.[10–14] De Felice has studied this same case for Schwarzschild and Kerr without decomposing the 4-force and using the angular velocity relative to the static (or distantly nonrotating) observers as his key variable.[18–22] Barrabès, Boisseau, and Israel[23] have done the same, but using the locally nonrotating observer relative velocity as the key variable.

It is worth noting that in the case of orthogonality of the vorticity vector and the relative velocity as occurs for the circular orbits the vanishing of the corotating Fermi-Walker relative curvature implies that the angular velocity-like quantity $\nu(U,u)/\rho_{(\mathrm{fw})}(u)$ of the center of relative curvature in the local rest space equals the magnitude of the vorticity.

9. Spatial gravitational forces

The spatial projection of the four-acceleration

$$a(U)^\alpha = DU^\alpha / d\tau_U \ , \tag{9.1}$$

when rescaled to take into account the differences in proper times, is the apparent three-acceleration as seen by the test observers

$$\begin{aligned} A(U,u)^\alpha &= \gamma(U,u)^{-1} P(u)^\alpha{}_\beta DU^\beta / d\tau_U \\ &= D_{(\mathrm{fw})}(U,u)[\gamma(U,u)u^\alpha + p(U,u)^\alpha]/d\tau_{(U,u)} \\ &= D_{(\mathrm{fw})}(U,u)p(U,u)^\alpha / d\tau_{(U,u)} - F^{(\mathrm{G})}_{(\mathrm{fw})}(U,u)^\alpha \ , \end{aligned} \tag{9.2}$$

and it can be rewritten in terms of the other two total spatial covariant derivatives in a single form

$$A(U,u)^\alpha = D_{(\mathrm{tem})}(U,u)p(U,u)^\alpha / d\tau_{(U,u)} - F^{(\mathrm{G})}_{(\mathrm{tem})}(U,u)^\alpha \tag{9.3}$$

where $\mathrm{tem} = \mathrm{fw}, \mathrm{cfw}, \mathrm{lie}$.

The spatial gravitational force (per unit mass)

$$\begin{aligned} F^{(\mathrm{G})}_{(\mathrm{tem})}(U,u)^\alpha &= \gamma(U,u) D_{(\mathrm{tem})}(U,u)u^\alpha / d\tau_{(U,u)} \\ &= \gamma(U,u)[g(u)^\alpha + H_{(\mathrm{tem})}(u)^\alpha{}_\beta \nu(U,u)^\beta] \ , \end{aligned} \tag{9.4}$$

is a Lorentz-like force determined by the gravitoelectric $g(u)^\alpha$ and gravitomagnetic $H_{(\mathrm{tem})}(u)^\alpha{}_\beta$ fields which in turn are simply related to the kinematical fields of the observer congruence

$$\begin{aligned} g(u)^\alpha &= -a(u)^\alpha \ , \\ H_{(\mathrm{fw})}(u)^\alpha{}_\beta &= \omega(u)^\alpha{}_\beta - \theta(u)^\alpha{}_\beta \ , \\ H_{(\mathrm{cfw})}(u)^\alpha{}_\beta &= 2\omega(u)^\alpha{}_\beta - \theta(u)^\alpha{}_\beta \ , \\ H_{(\mathrm{lie})}(u)^\alpha{}_\beta &= 2\omega(u)^\alpha{}_\beta - 2\theta(u)^\alpha{}_\beta \ . \end{aligned} \tag{9.5}$$

It is useful to introduce a single gravitomagnetic vector field which determines the antisymmetric part of all of the various gravitomagnetic tensor fields

$$H(u)^\alpha = 2\omega(u)^\alpha \ , \tag{9.6}$$

namely so that

$$2\omega(u)^\alpha{}_\beta \nu(U,u)^\beta = [\nu(U,u) \times_u \vec{H}(u)]^\alpha \ . \tag{9.7}$$

The symmetric part of the gravitomagnetic tensor field is just a multiple of the expansion tensor for each case.

If the acceleration of the test particle equals a spacetime force $a(U)^\alpha = f(U)^\alpha$ and one introduces the rescaled spatial projection

$$F(U,u)^\alpha = \gamma(U,u)^{-1} P(u)^\alpha{}_\beta f(U)^\beta \ , \tag{9.8}$$

then the spatial projection of the force equation ("spatial equation of motion") may be written in the form

$$D_{(\mathrm{tem})}(U,u)p(U,u)^\alpha/d\tau_{(U,u)} = F^{(\mathrm{G})}_{(\mathrm{tem})}(U,u)^\alpha + F(U,u)^\alpha \ . \tag{9.9}$$

The spatial gravitational force represents the combined inertial forces due to the motion of the family of test observers. It arises in the same way as the noninertial forces in nonrelativistic mechanics, namely as a part of the total acceleration which is moved to the opposite side of the "acceleration equals force per unit mass" equation with a sign change.

Space curvature effects are encoded in the total spatial covariant derivative itself. Suppose one considers a world line segment which starts and ends on a single observer world line, and one transports a spatial vector along both paths (general world line and observer world line) from the initial to the final point using the transport associated with one of the three kinds of total spatial covariant derivatives. In each case the two final vectors will have distinct directions, and in the Lie case, different magnitudes in general, due to curvature effects associated with the spatial metric, similar to the case of two such paths in the simpler case of a fixed Riemannian three-manifold. For a timelike Killing vector field test observer congruence in a stationary spacetime, where the Lie and corotating total spatial covariant derivatives coincide, the corotating Fermi-Walker space curvature effect is exactly that due to the curvature of the natural projected Riemannian metric on the quotient space of observer world lines. For circular orbits in a stationary axisymmetric spacetime, this effect may be calculated explicitly using the tangent cone to the embedding of the plane of the orbit, as discussed in appendix 1.A of Arnold.[33]

Since index shifting does not commute with the Lie total spatial covariant derivative, letting this derivative act instead on $p(U,u)_\alpha$ leads to an additional expansion term in the Lie spatial gravitational force. Conveniently introducing a "flattened" Lie total spatial covariant derivative by

$$D_{(\mathrm{lie}\flat)}(U,u)X^\alpha/d\lambda = P(u)^{\alpha\beta} D_{(\mathrm{lie}\flat)}(U,u)X_\beta/d\lambda \ , \tag{9.10}$$

and a "flattened" Lie spatial gravitational force with a corresponding gravitomagnetic field

$$H_{(\mathrm{lie}\flat)}(u)^\alpha{}_\beta = 2\omega(u)^\alpha{}_\beta \ , \tag{9.11}$$

one has the analogous form of the force equation

$$D_{(\mathrm{lie}\flat)}(U,u)p(U,u)^\alpha/d\tau_{(U,u)} = F^{(\mathrm{G})}_{(\mathrm{lie}\flat)}(U,u)^\alpha + F(U,u)^\alpha \ . \tag{9.12}$$

This notation facilitates the comparison of the different choices without requiring index shifting.

10. Massless test particles

Consider a massless test particle following a null path with affine parameter λ_P and tangent P^α locally expressible as $dx^\alpha/d\lambda_P$ in terms of local coordinates.

Interpreting P^α as the 4-momentum directly, rather than the 4-momentum per unit mass of the previous discussion for a massive test particle, one may essentially make the substitution $(U^\alpha, \tau_U, \gamma(U,u), f^\alpha, F^\alpha) \to (P^\alpha, \lambda_P, E(P,u), f^\alpha, F^\alpha)$ in that discussion to reinterpret the results and formulas in the new context. Since the speed now satisfies

$$\nu(U,u) = d\ell_{(U,u)}/d\tau_{(U,u)} = 1 \ , \tag{10.1}$$

the relative velocity is a unit vector $\hat\nu(P,u)^\alpha = p(P,u)^\alpha/E(P,u)$, while the energy $E(P,u)$ and magnitude of the spatial momentum $p(P,u)^\alpha$ are now related by $E(P,u) = p(P,u)$. The spatial equation of motion then becomes simply

$$D_{(\rm tem)}(U,u) p(U,u)^\alpha / d\tau_{(U,u)} = F^{(\rm G)}_{(\rm tem)}(U,u)^\alpha + F(P,u)^\alpha \ , \tag{10.2}$$

where the spatial gravitational force is

$$F^{(\rm G)}_{(\rm tem)}(U,u)^\alpha = E(P,u)[g(u)^\alpha + H_{(\rm tem)}(u)^\alpha{}_\beta \nu(U,u)^\beta] \ . \tag{10.3}$$

For null geodesics the (affine-parameter-dependent) forces $f(P)^\alpha$ and $F(P,u)^\alpha$ are zero, but accelerated photon motion is important in the discussion of certain relativistic phenomena like the Sagnac effect, where photons traveling in opposite directions around a loop via mirrors or fiber optics are indeed accelerated.

For a massless test particle, the ordinary and corotating Fermi-Walker tangential relative accelerations are automatically zero since the relative velocity is a unit vector, and the relative centripetal acceleration has exactly the same form as for the nonzero mass case

$$a^{(\perp)}_{(\rm tem)}(P,u) = \nu(P,u)^2/\rho_{(\rm tem)}(P,u) = 1/\rho_{(\rm tem)}(P,u) \ , \quad {\rm tem=fw,cfw} \ . \tag{10.4}$$

This acceleration vanishes for null trajectories which undergo "relatively straight" relative motion.

11. Observer-adapted spatial frames

The projection formalism is greatly simplified if expressed in terms of a spacetime frame adapted to the splitting of each tangent space defined by the test observer family. Let $\{E_a{}^\alpha\}$ be a spatial frame, i.e., such that it is a basis of each local rest space LRS_u. It is convenient to express the above results in terms of the observer-adapted spacetime frame $\{u^\alpha, E_a{}^\alpha\}$, with dual frame $\{-u_\alpha, W^a{}_\alpha\}$. Spatial fields then only have spatially-indexed frame components nonzero, like the spatial metric $h_{ab} = P(u)_{ab}$. Note that observer-adapted spatial frame components are distinct from the Latin-indexed coordinate components; unless otherwise indicated Latin indices in formulas will refer to the frame components.

Let the frame derivatives of functions be denoted by the comma notation

$$u(f) = f_{,0} \ , \quad E_a{}^\alpha \partial_\alpha f = f_{,a} \ . \tag{11.1}$$

To express derivatives of tensor fields, one needs the temporal and spatial derivatives of the spatial frame vectors themselves, as well as their Lie brackets

$$\begin{aligned} \nabla_{(\text{tem})}(u)E_a{}^\alpha &= C_{(\text{tem})}(u)^b{}_a E_b{}^\alpha \ , \qquad \nabla(u)_{E_a} E_b{}^\alpha = \Gamma(u)^c{}_{ab} E_c{}^\alpha \ , \\ (P(u)[E_a, E_b])^\alpha &= C(u)^c{}_{ab} E_c{}^\alpha \ , \end{aligned} \tag{11.2}$$

where

$$C_{(\text{cfw})}(u)^a{}_b = C_{(\text{fw})}(u)^a{}_b + \omega(u)^a{}_b = C_{(\text{lie})}(u)^a{}_b + \theta(u)^a{}_b \ . \tag{11.3}$$

For example, if $X^\alpha = X^a E_a{}^\alpha$ is a spatial vector field, one has

$$\nabla_{(\text{tem})}(u)X^a = X^a{}_{,0} + C_{(\text{tem})}(u)^a{}_b X^b \ , \quad \nabla(u)_b X^a = X^a{}_{,b} + \Gamma(u)^a{}_{bc} X^c \ , \tag{11.4}$$

while if it is only defined along the world line, one has

$$D_{(\text{tem})}(U,u)X^a/d\tau_{(U,u)} = dX^a/d\tau_{(U,u)} + C_{(\text{tem})}(u)^a{}_b X^b + \Gamma(u)^a{}_{bc}\nu(U,u)^b X^c \ . \tag{11.5}$$

The frame components of the spatial connection are easily expressed in terms of the spatial metric derivatives and the spatial structure functions of the spatial frame

$$\Gamma(u)_{abc} = \tfrac{1}{2}(h_{\{ab,c\}_-} + C(u)_{\{abc\}_-}) \ , \tag{11.6}$$

where $A_{\{abc\}_-} = A_{abc} - A_{bca} + A_{cab}$.

For an orthonormal frame, the Fermi-Walker and corotating Fermi-Walker frame coefficients $C_{(\text{tem})}(u)^a{}_b$ are antisymmetric. For the special Serret-Frenet orthonormal spatial frame associated with the observer congruence, only two independent such coefficients exist; in the Fermi-Walker case, since they arise from the spatial projection of the Fermi-Walker derivatives of the frame vectors along the observer world lines, they are just the first and second torsions of those world lines.[28] (Note that according to the definition,[28] a single trajectory of a nontrivial quasi-Killing vector field is a Killing vector field trajectory, but the family of such trajectories for a single quasi-Killing vector field consists of trajectories not of one single Killing vector field.)

12. Spatial gravito-potentials

The discussion of the measurement by a congruence of test observers of tensor fields and of tensor differential equations using no other spacetime structure may be referred to as the congruence point of view. (Original references for spacetime splittings are given elsewhere.[1]) While in a generalized sense the 4-velocity u^α serves as a 4-vector potential for the gravitoelectric and gravitomagnetic vector force fields in this point of view (a partial splitting of spacetime), scalar and spatial vector potentials analogous to those in electromagnetism may be defined only by introducing certain equivalence classes of local coordinates which are adapted to the congruence of observer world lines in some way (a full splitting of spacetime). For a general congruence with nonzero vorticity, the appropriate coordinates are comoving, leading to an approach called the threading point of view, which merely represents the general discussion of observer measured quantities in an adapted coordinate system. For a special case of a vorticity-free congruence where the gravitomagnetic vector field vanishes, it is more natural to refer to the partial splitting as the hypersurface point of view, and for the full splitting one can use more general coordinates adapted to the family of orthogonal hypersurfaces admitted by the congruence (the time coordinate hypersurfaces), allowing the time coordinate lines to be determined by a second independent congruence. One then has the choice of representing all the hypersurface-forming observer-measured quantities directly (the hypersurface point of view) or of working in a hybrid two-congruence approach called the slicing point of view, in which the evolution is described in terms of the second congruence. This latter approach is the one well known from the work of Arnowit, Deser, and Misner.[34,35] The gravitomagnetic vector field reappears in this latter approach as a relative velocity effect introduced through the use of a new temporal derivative along the time lines rather than along the observer world lines.[36] Thus for a given spacelike slicing and timelike threading (together forming a "nonlinear reference frame"), one obtains two distinct families of observers and three distinct points of view which agree only when the slicing and threading are orthogonal.

Let $\{x^\alpha\} = \{t, x^a\}$ with $x^0 = t$ be a set of local coordinates (said to be "adapted to the nonlinear reference frame") for which the time coordinate hypersurfaces (constant t) belong to the given slicing of spacetime and the time coordinate lines (constant x^a) belong to the given threading congruence. Introduce also the vector field tangent to the time coordinate lines $e_0{}^\alpha = \delta^\alpha{}_0$.

12.1. The threading point of view

First consider the threading point of view, which is especially useful in a stationary spacetime where it enables one to interpret the spacetime geometry in terms of the quotient space geometry on the space of Killing observers. The adapted coordinates are comoving coordinates for the 4-velocity of the observer congruence $u^\alpha = m^\alpha = M^{-1}\delta^\alpha{}_0$, and the observer world lines coincide with the time coordinate lines.

The spacetime line element (covariant metric) takes the form

$$\begin{aligned} ds^2 &= -M^2(dt - M_a dx^a)^2 + \gamma_{ab} dx^a dx^b \ , \\ &= M^2[-(dt - M_a dx^a)^2 + \tilde{\gamma}_{ab} dx^a dx^b] \ , \end{aligned} \tag{12.1}$$

where γ_{ab} parametrizes the spatial metric and is the matrix inverse of $\gamma^{ab} = P(m)^{ab}$, while $\tilde{\gamma}_{ab} = M^{-2}\gamma_{ab}$ parametrizes the "optical spatial metric" $\tilde{P}(m)_{\alpha\beta} = M^{-2}P(m)_{\alpha\beta}$ obtained by a conformal rescaling of the spatial metric, using the terminology of Abramowicz et al.[3] The "spatial derivatives" $\epsilon_a = \partial/\partial x^a + M_a \partial/\partial t = \epsilon_a{}^\alpha \partial/\partial x^\alpha$ define the basis vector fields $\{\epsilon_a{}^\alpha\}$ of the local rest space of the test observers (which together with m^α form an observer-adapted frame for which

$C_{(\mathrm{lie})}(m)^a{}_b = 0$ and $C(m)^a{}_{bc} = 0$), and the matrix γ_{ab} is the matrix of their inner products. The associated components of the spatial connection

$$\epsilon_a{}^\alpha \nabla_\alpha(m) \epsilon_b{}^\beta = \Gamma(m)^c{}_{ab} \epsilon_c{}^\beta \ , \tag{12.2}$$

expressable as

$$\Gamma(m)^c{}_{ab} = \tfrac{1}{2} \gamma^{cd} (\gamma_{da,b} - \gamma_{ab,d} + \gamma_{bd,a}) \ , \tag{12.3}$$

where $f_{,a} = \epsilon_a{}^\alpha \partial f / \partial x^\alpha$, may be used to evaluate the spatial covariant derivative of a spatial vector field $X^\alpha = X^a \epsilon_a{}^\alpha$ (parametrized by its spatially-indexed contravariant coordinate components X^a) entirely in terms of the components in that frame in the usual way.[1] Analogous tilde expressions hold for the components $\tilde{\Gamma}(m)^c{}_{ab}$ of the optical connection $\tilde{\nabla}(m)_\alpha$.

The shift 1-form $M_\alpha = M_a \delta^a{}_\alpha$ is a spatial 1-form which determines the shift of the orientation of the local rest spaces of the test observers away from the coordinate time hypersurfaces, while the lapse function M relates coordinate time along the time lines to the test observer proper time. The combination $\nu(n,m)^\alpha = M M^a \delta^\alpha{}_a$ is the relative velocity field of the normal trajectories to the coordinate time hypersurfaces. The lapse and shift serve as scalar and vector potentials for the gravitoelectric and gravitomagnetic vector fields of the test observer congruence, while the spatial metric generates the symmetric part of the gravitomagnetic tensor field, which is proportional to the expansion tensor

$$\begin{aligned} g(m)_\alpha &= -a(m)_\alpha = -\nabla(m)_\alpha \ln M - \pounds(m)_{e_0} M_\alpha \ , \\ H(m)^\alpha &= 2\omega(m)^\alpha = M \eta(m)^{\alpha\beta\gamma} \nabla(m)_\beta M_\gamma = M[\nabla(m) \times_m \vec{M}]^\alpha \ , \\ \theta(m)_{\alpha\beta} &= \tfrac{1}{2} \pounds(m)_{e_0} P(m)_{\alpha\beta} \ . \end{aligned} \tag{12.4}$$

In the observer-adapted frame only the spatially indexed components (distinct from the coordinate components of this type) of these fields are nonzero, and the Lie derivatives reduce to the partial derivatives of these components with respect to the t coordinate, for example

$$\theta(m)_{ab} = \tfrac{1}{2} \partial_t \gamma_{ab} \ . \tag{12.5}$$

The total spatial covariant derivative of the spatial momentum becomes explicitly

$$\begin{aligned} & D_{(\mathrm{tem})}(U,m) p(U,m)^a / d\tau_{(U,m)} \\ & = dp(U,m)^a / d\tau_{(U,m)} + \gamma(U,m) C_{(\mathrm{tem})}(m)^a{}_b \nu(U,m)^b - F^{(\mathrm{SC})}(U,m)^a \ , \end{aligned} \tag{12.6}$$

where

$$F^{(\mathrm{SC})}(U,m)^a = -\gamma(U,m) \Gamma(m)^a{}_{bc} \nu(U,m)^b \nu(U,m)^c \tag{12.7}$$

defines the "space curvature" force in the threading point of view.

12.2. The hypersurface and slicing points of view

In the hypersurface and slicing points of view, given a family of spacelike hypersurfaces with unit normal n^α, the spacetime line element in adapted local coordinates takes the form

$$\begin{aligned} ds^2 &= -N^2 dt^2 + g_{ab}(dx^a + N^a dt)(dx^b + N^b dt) \\ &= N^2[-dt^2 + \tilde{g}_{ab}(dx^a + N^a dt)(dx^b + N^b dt)] \ . \end{aligned} \tag{12.8}$$

Here $g_{ab} = P(n)_{ab}$ parametrizes the spatial metric and is the matrix inverse of g^{ab}, while $\tilde{g}_{ab} = N^{-2} g_{ab}$ parametrizes the "optical spatial metric" $\tilde{P}(n)_{\alpha\beta} = N^{-2} P(n)_{\alpha\beta}$. The spatial coordinate vectors themselves $e_a{}^\alpha = \delta^\alpha{}_a$ form a spatial frame (which together with n^α form an observer-adapted frame with $C(n)^a{}_{bc} = 0$ and $C_{(\mathrm{lie})}(n)^a{}_b = N^{-1}\partial N^a/\partial x^b$) which may be used to express the spatial covariant derivative of a spatial vector field $X^\alpha = X_b g^{ba} e_a{}^\alpha$ (parametrized by its spatially-indexed covariant coordinate components X_a) in terms of the associated connection components, namely

$$e_a{}^\alpha \nabla_\alpha(n) e_b{}^\beta = \Gamma(n)^c{}_{ab} e_c{}^\gamma \ , \tag{12.9}$$

expressable as

$$\Gamma(n)^c{}_{ab} = \tfrac{1}{2} g^{cd}(g_{da,b} - g_{ab,d} + g_{bd,a}) \ , \tag{12.10}$$

where here $f_{,a} = \partial f/\partial x^a$. Analogous tilde expressions hold for the components $\tilde{\Gamma}(n)^c{}_{ab}$ of the optical connection $\tilde{\nabla}(n)_\alpha$.

The 4-velocity of the associated test observers is the unit normal

$$n^\alpha = N^{-1}[\delta^\alpha{}_0 - N^a \delta^\alpha{}_a] \ , \tag{12.11}$$

and their world lines are the orthogonal trajectories to the time hypersurfaces. The shift vector field $N^\alpha = N^a \delta^\alpha{}_a$ is a spatial vector field which determines the shift of the time lines away from these orthogonal trajectories, while the lapse function N relates the coordinate time along the observer world lines to the observer proper time. The combination $\nu(e_0, n)^\alpha = N^{-1} N^a \delta^\alpha{}_a$ is the relative velocity field of the threading curves.

The distinguishing feature of the slicing point of view compared to the hypersurface point of view or the threading point of view for the same congruence of observers (the latter requiring a comoving coordinate system) is that it uses a new Lie temporal derivative along the threading curves rather than along the observer world lines

$$\nabla_{(\mathrm{lie})}(n, e_0) = N^{-1} \pounds(n)_{e_0} = \nabla_{(\mathrm{lie})}(n) + N^{-1} \pounds(n)_{\vec{N}} \ . \tag{12.12}$$

This in turn leads to a new Lie spatial total covariant derivative along a test particle worldline

$$\begin{aligned} D_{(\mathrm{lie})}(U, n, e_0) X^\alpha / d\tau_{(U,n)} &= [\nabla_{(\mathrm{lie})}(n, e_0) + \nu(U,n)^\beta \nabla(n)_\beta] X^\alpha \\ &= D_{(\mathrm{lie})}(U, n) X^\alpha / d\tau_{(U,n)} - [N^{-1} \nabla(n)_\beta N^\alpha] X^\beta \ . \end{aligned} \tag{12.13}$$

Expressing the two Lie total covariant derivatives in the observer-adapted frame leads to

$$\begin{aligned} D_{(\mathrm{lie})}(U,n)X^a/d\tau_{(U,n)} &= dX^a/d\tau_{(U,n)} + \Gamma(n)^a{}_{bc}\nu(U,n)^b X^c + X^b N^{-1}\partial N^a/\partial x^b \\ &= dX^a/d\tau_{(U,n)} + \Gamma(n)^a{}_{bc}[\nu(U,n)^b - \nu(e_0,n)^b]X^c \\ &\quad + X^b N^{-1}\nabla(n)_b N^a \\ &= D_{(\mathrm{lie})}(U,n,e_0)X^a/d\tau_{(U,n)} + X^b N^{-1}\nabla(n)_b N^a \ . \end{aligned} \tag{12.14}$$

When $X^a = \nu(U,n)^a$, these equations define the slicing Lie relative acceleration $a_{(\mathrm{lie})}(U,n,e_0)^\alpha$ and its relation to the hypersurface quantity, namely

$$a_{(\mathrm{lie})}(U,n,e_0)^a - a_{(\mathrm{lie})}(U,n)^a = -\nu(U,n)^b N^{-1}\nabla(n)_b N^a \ . \tag{12.15}$$

The spatial equation of motion of a test particle then takes the form

$$D_{(\mathrm{lie})}(U,n)p(U,n)^\alpha/d\tau_{(U,n)} = \gamma(U,n)[g(n)^\alpha + H_{(\mathrm{lie})}(n)^\alpha{}_\beta \nu(U,n)^\beta] + F(U,n)_\alpha \tag{12.16}$$

in the hypersurface point of view and

$$\begin{aligned} D_{(\mathrm{lie})}(U,n,e_0)p(U,n)^\alpha/d\tau_{(U,n)} &= \gamma(U,n)[g(n)^\alpha + H_{(\mathrm{lie})}(n,e_0)^\alpha{}_\beta \nu(U,n)^\beta] \\ &\quad + F(U,n)^\alpha \ , \\ D_{(\mathrm{lie})}(U,n,e_0)p(U,n)_\alpha/d\tau_{(U,n)} &= \gamma(U,n)[g(n)_\alpha + H_{(\mathrm{lie}\flat)}(n,e_0)_{\alpha\beta}\nu(U,n)^\beta] \\ &\quad + F(U,n)_\alpha \end{aligned} \tag{12.17}$$

in the slicing point of view, where the hypersurface Lie gravitomagnetic tensor field is just minus twice the expansion tensor when the equations of motion are expressed in contravariant form, but zero in the covariant form

$$\begin{aligned} H_{(\mathrm{lie})}(n)_{\alpha\beta} &= N^{-1}[2\nabla(n)_{(\alpha}N_{\beta)} - \pounds(n)_{e_0}P(n)_{\alpha\beta}] = -2\theta(n)_{\alpha\beta} \ , \\ H_{(\mathrm{lie}\flat)}(n)_{\alpha\beta} &= 0 \end{aligned} \tag{12.18}$$

and the slicing gravitomagnetic tensor fields are

$$\begin{aligned} H_{(\mathrm{lie})}(n,e_0)_{\alpha\beta} &= N^{-1}[\nabla(n)_\alpha N_\beta - \pounds(n)_{e_0}P(n)_{\alpha\beta}] \ , \\ H_{(\mathrm{lie}\flat)}(n,e_0)_{\alpha\beta} &= N^{-1}\nabla(n)_\alpha N_\beta \ . \end{aligned} \tag{12.19}$$

The contravariant and covariant forms of the equation of motion differ by a term arising from the Lie temporal derivative of the spatial metric. For the contravariant form of the equation of motion the symmetric part of the gravitomagnetic tensor differs from $-2\theta(n)_{\alpha\beta}$ by the missing reversed-index shift derivative term which would symmetrize the term which is present. This missing term now contributes by its absense to the slicing gravitomagnetic vector field but is restored by an extra shift derivative term in the corresponding second order acceleration equation due to the relative velocity of the observers and evolution curves, thus restoring the analogy with the threading point of view at that level.[1]

The lapse and shift serve as scalar and vector potentials for the gravitoelectric and gravitomagnetic vector fields in the slicing point of view. The gravitoelectric field is still the sign-reversed acceleration of the test observers in both points of view, but the slicing gravitomagnetic vector field is now a result of the relative motion of the time lines with respect to the observer world lines

$$\begin{aligned} g(n)_\alpha &= -a(n)_\alpha = -\nabla(n)_\alpha \ln N \ , \qquad H_{(\mathrm{lie})}(n)^\alpha = 0 \ , \\ H_{(\mathrm{lie})}(n,e_0)^\alpha &= N^{-1}\eta(n)^{\alpha\beta\gamma}\nabla(n)_\beta N_\gamma = N^{-1}[\nabla(n)\times_n \vec{N}]^\alpha \ , \end{aligned} \tag{12.20}$$

In terms of the gravitomagnetic vector field, the slicing spatial equation of motion takes the form

$$\begin{aligned} D_{(\mathrm{lie})}(U,n,e_0)p(U,n)^\alpha/d\tau_n &= \gamma(U,n)\{g(n)^\alpha + \tfrac{1}{2}[\nu(U,n)\times_n \vec{H}_{(\mathrm{lie})}(n,e_0)]^\alpha \\ &\quad + H^{(\mathrm{SYM})}_{(\mathrm{lie})}(n,e_0)^\alpha{}_\beta \nu(U,n)^\beta\} \ . \end{aligned} \tag{12.21}$$

All of these expressions have simple analogous forms when expressed in the observer-adapted frame. The total spatial covariant derivative of the spatial momentum becomes explicitly

$$\begin{aligned} D_{(\mathrm{lie})}(U,n)p(U,n)^a/d\tau_{(U,n)} &= dp(U,n)^a/d\tau_{(U,n)} - F^{(\mathrm{SC})}(U,n)^a \\ &\quad + p(U,n)^b \partial N^a/\partial x^b \ , \\ D_{(\mathrm{lie})}(U,n,e_0)p(U,n)^a/d\tau_{(U,n)} &= dp(U,n)^a/d\tau_{(U,n)} - F^{(\mathrm{SC})}(U,n,e_0)^a \end{aligned} \tag{12.22}$$

where

$$\begin{aligned} F^{(\mathrm{SC})}(U,n)^a &= -\gamma(U,n)\Gamma(n)^a{}_{bc}\nu(U,n)^b\nu(U,n)^c \ , \\ F^{(\mathrm{SC})}(U,n,e_0)^a &= -\gamma(U,n)\Gamma(n)^a{}_{bc}[\nu(U,n)^b - N^{-1}N^b]\nu(U,n)^c \end{aligned} \tag{12.23}$$

defines the “space curvature” force in the two points of view.

12.3. Stationary spacetimes

Suppose ${e_0}^\alpha$ (and therefore m) is a timelike Killing vector field in some open submanifold of a stationary spacetime, implying that $\theta(m)_{\alpha\beta} = 0$, and suppose that n^α is also timelike in some other open submanifold overlapping with the first. One then has a nonlinear reference frame adapted to the stationarity in which one may consider threading, slicing, and hypersurface points of view. The Kerr black hole spacetimes in Boyer-Lindquist coordinates which are adapted to the distantly nonrotating (static) observers determining the threading congruence and to the locally nonrotating observers determining the slicing congruence are a good example to keep in mind.

All Lie derivatives of stationary fields along ${e_0}^\alpha$ vanish. This eliminates the complication of the mixing of time coordinate derivatives with spatial coordinate derivatives in the spatial derivatives of component fields in threading point of view when differentiating stationary tensor fields, since their various components are independent of the time coordinate. In all cases the spatial metric projects to a Riemannian metric on the threading quotient space where the relative motion takes place and all calculations become much more straightforward. The scalar and vector potentials then become potentials in the usual sense on this space since spatial projection is unnecessary and the curl and gradient are just the ones on this Riemannian observer quotient space. Furthermore the threading observer expansion tensor vanishes although the slicing observer one does not, leading to the coincidence of the Lie and corotating Fermi-Walker derivatives in the threading point of view. The temporal derivative observer-adapted frame structure functions in the threading point of view therefore satisfy $C_{(\mathrm{cfw})}(m)^a{}_b = C_{(\mathrm{lie})}(m)^a{}_b = 0$, and the two associated total spatial covariant derivatives coincide as well. In the case of the slicing observer-adapted frame, one instead has

$$C(n)_{(\mathrm{cfw})}{}^a{}_b - \theta(n)^a{}_b = C(n)_{(\mathrm{lie})}{}^a{}_b = N^{-1}\partial N^a/\partial x^b \ , \qquad (12.24)$$

but for relative motion along a spatial Killing direction $\nu(U,n)^b\partial N^a/\partial x^b = 0$, so the shift derivative term does not contribute to the relative curvature and the centripetal acceleration in that case, making the Lie relative curvature the simplest such curvature in the hypersurface point of view. However, the expansion term does lead to an additional term in the corotating Fermi-Walker centripetal acceleration which is linear in the relative velocity; a similar linear term arises also in the the slicing point of view expression due to the relative velocity of the observers themselves relative to the threading.

13. Spatial coordinate line curvature in stationary spacetimes with additional symmetry

Suppose $e_0{}^\alpha$ is a timelike Killing vector field in a stationary spacetime with an additional spacelike Killing vector field for which t and x^3 respectively are comoving coordinates, so that the metric only depends on the remaining two coordinates. Assume also that the spatial coordinates are orthogonal, a situation which describes many interesting stationary spacetimes, including rotating black holes (Kerr spacetimes) in Boyer-Lindquist coordinates and the Gödel spacetime in the usual Cartesian-like or cylindrical-like coordinates.

Consider a test particle worldline following a Killing trajectory in the 2-surface of the coordinates t and x^3 with the remaining two coordinates fixed, implying that in either the associated threading or slicing point of view, the unit velocity vector has a single nonvanishing constant spatial coordinate component. Constant speed circular orbits in the above-mentioned spacetimes are of this type, for example.

Consider the threading point of view, for example, where $\hat\nu(U,m)^a = \gamma_{33}{}^{-1/2}\delta^a{}_3$ and

$$D_{(\mathrm{lie})}(U,m)\hat\nu(U,m)^a/d\ell_{(U,m)} = \Gamma(m)^a{}_{33}(\hat\nu(U,m)^3)^2 = -\gamma_{aa}{}^{-1}\partial(\ln\gamma_{33}{}^{1/2})/\partial x^a \ , \qquad (13.1)$$

or, since the spatial coordinates are orthogonal, in terms of the associated orthonormal ("physical") components indicated with the "hat" notation

$$D_{(\rm lie)}(U,m)\hat{\nu}(U,m)^{\hat{a}}/d\ell_{(U,m)} = -{\gamma_{aa}}^{-1/2}\partial(\ln {\gamma_{33}}^{1/2})/\partial x^a = \kappa(3,m)^{\hat{a}} \ , \qquad (13.2)$$

which can only be nonzero for $a = 1,2$. The Lie relative curvature $\kappa_{(\rm lie)}(U,m)$ will just be the square root of the sum of the squares of the these two physical components; in fact for circular orbits only one of these will be nonzero. (For the case of additional symmetry like stationary cylindrically symmetric spacetimes of which the Gödel spacetime is an example, one can evaluate this spatial curvature for each spatial coordinate line along a Killing vector field, with the result for a general timelike Killing trajectory being obtained in a simple way from the individual results, i.e., as for a helical Killing trajectory in Gödel.) The corresponding optical relative curvature has the same formula with the tilde metric and derivative, the latter of which is described in the appendix. A similar formula holds for the physical observer-adapted frame components in the hypersurface point of view

$$D_{(\rm lie)}(U,n)\hat{\nu}(U,n)^{\hat{a}}/d\ell_{(U,n)} = -{g_{aa}}^{-1/2}\partial(\ln {g_{33}}^{1/2})/\partial x^a = \kappa(3,n)^{\hat{a}} \ , \qquad (13.3)$$

and for its optical generalization. All formulas continue to hold for massless test particle trajectories as well.

These coordinate curvature quantities $\kappa(3,u)^{\hat{a}}$ will be referred to as the signed Lie relative curvatures ("signed" since they may take all real values). In each of these four formulas, the relative curvature of these special trajectories corresponds exactly to the coordinate line curvature in the quotient space Riemannian geometry or its optical version for the relevant observer point of view (threading, slicing, or hypersurface). In the static case, all of these points of view coincide and only two distinct relative Lie curvatures exist.

14. Circular orbits in stationary axisymmetric spacetimes

Now specialize to the case of a stationary axisymmetric spacetime with coordinates $\{t,z,\rho,\phi\}$, where t and ϕ are comoving with respect to the Killing vector fields associated with the stationary and axial symmetries respectively, and the latter three coordinates are orthogonal, as occurs for the stationary cylindrically symmetric Gödel and rotating Minkowski spacetimes. Only the single threading observer-adapted spatial frame vector ${\varepsilon_\phi}^\alpha = {\delta^\alpha}_\phi + M_\phi {\delta^\alpha}_t$ differs from the orthogonal spatial coordinate frame vectors.

For circular orbits, the observer-adapted physical components of the relative velocity along the angular direction are

$$\begin{aligned} \nu(U,m)^{\hat{\phi}} &= M^{-1}{\gamma_{\phi\phi}}^{1/2}\dot{\phi}/(1-M_\phi\dot{\phi}) \ , \\ \nu(U,n)^{\hat{\phi}} &= N^{-1}{g_{\phi\phi}}^{1/2}(\dot{\phi}+N^\phi) \ , \end{aligned} \qquad (14.1)$$

while

$$\nu(U,n)^{\hat{\phi}} - \nu(e_0,n)^{\hat{\phi}} = N^{-1}{g_{\phi\phi}}^{1/2}\dot{\phi} \ , \qquad (14.2)$$

where the coordinate angular velocity $\dot\phi = d\phi/dt$ along the test particle world line is constant for the stationary circular orbits of constant speed to be considered here. Thus the test particle world line is a Killing trajectory and the ordinary time derivative terms vanish in the total spatial covariant derivatives along the test world line.

Attention will be confined to those circular orbits for which the z-derivative relative curvature expression vanishes

$$\begin{aligned} \kappa(\phi,m)^{\hat z} &= \kappa(\phi,n)^{\hat z} = 0 \ ,\\ \kappa(\phi,m)^{\hat\rho} &= -\gamma_{\rho\rho}{}^{-1/2}\partial(\ln\gamma_{\phi\phi}^{1/2})/\partial\rho \ ,\\ \kappa(\phi,n)^{\hat\rho} &= -g_{\rho\rho}{}^{-1/2}\partial(\ln g_{\phi\phi}^{1/2})/\partial\rho \ . \end{aligned} \tag{14.3}$$

This limits one to the equatorial circular orbits in the Kerr spacetime. The Lie relative centripetal acceleration then has the single nonzero physical observer-adapted frame component

$$a_{(\rm lie)}(U,u)^{(\perp)\hat\rho} = \kappa(\phi,u)^{\hat\rho}|\nu(U,u)^{\hat\phi}|^2 \ , \quad u=m,n \ . \tag{14.4}$$

Since $\theta(n)_{\phi\phi}=0$, the hypersurface point of view Lie relative acceleration is fortuitously orthogonal to the relative velocity and so directly represents the relative centripetal acceleration.

The nonzero observer-adapted physical components of the various fields are

$$\begin{aligned} g(m)_{\hat\rho} &= -(\gamma_{\rho\rho})^{-1/2}(\ln M)_{,\rho} \ , \qquad H(m)^{\hat z} = M(\gamma_{\rho\rho}\gamma_{\phi\phi})^{-1/2}M_{\phi,\rho} \ ,\\ g(n)_{\hat\rho} &= -(g_{\rho\rho})^{-1/2}(\ln N)_{,\rho} \ , H(n,e_0)^{\hat z} = N^{-1}(g_{\rho\rho}g_{\phi\phi})^{-1/2}N_{\phi,\rho} \ , \end{aligned} \tag{14.5}$$

and

$$\begin{aligned} H(n,e_0)_{(\hat\rho\hat\phi)} &= \tfrac12 H(n)_{\hat\rho\hat\phi} = -\theta(n)_{\hat\rho\hat\phi} = \tfrac12 N^{-1}(g_{\phi\phi}/g_{\rho\rho})^{1/2}N^\phi{}_{,\rho}\\ &= \tfrac12 C_{(\rm lie)}(n)^{\hat\phi}{}_{\hat\rho} = \tfrac12 C_{(\rm cfw)}(n)^{\hat\phi}{}_{\hat\rho} \ , \end{aligned} \tag{14.6}$$

where the comma indicates the coordinate partial derivative here.

The physical 4-force responsible for the motion of the test particle on the circular orbit is related to the relative force by equation (9.8). Since the projection $P(u)$ acts as the identity in the radial direction orthogonal to the relative motion, one has the simpler relationship between the 4-force, the 4-acceleration, and the relative forces

$$f(U)^{\hat\rho} = \gamma(U,u)F(U,u)^{\hat\rho} = \gamma(U,u)[-F^{(\rm SC)}(U,u)^{\hat\rho} - F^{(\rm G)}(U,u)^{\hat\rho}] = a(U)^{\hat\rho} \ . \tag{14.7}$$

The middle equality is the equation of motion for the circular orbit, while the last equality is the value of the acceleration of the orbit. The same considerations apply to circular orbits in the equatorial plane of the Kerr spacetime and its Schwarzschild limit in terms of the usual "spherical" radial coordinate in that plane.

Iyer and Vishveshwara have given the complete Serret-Frenet frame formulas for the stationary axisymmetric Killing trajectories (arbitrary constant speed circular orbits) in Kerr, Gödel, Minkowski and several other spacetimes.[28] These depend only on the single ratio of the two Killing vector components of the tangent vector $\delta^\alpha{}_t + \omega\delta^\alpha{}_\phi$, or equivalently of $\omega^{-1}\delta^\alpha{}_t + \delta^\alpha{}_\phi$ because of its normalization. The limit $\omega^{-1} = 0$ of their formula for the curvature κ of the associated Killing trajectory gives exactly $|\kappa(\phi, n)^{\hat\rho}|$, with their frame vectors reducing to $(e^\alpha_{(0)}, e^\alpha_{(1)}, e^\alpha_{(2)}) = (-e^\alpha_{\hat\phi}, e^\alpha_{\hat\rho}, -n^\alpha)$.

15. The Abramowicz et al Approach Demystified

By introducing the optical metric in the threading or slicing points of view, the full spacetime metric is conformal to one with unit lapse and the same shift 1-form or vector field respectively, the overall conformal factor being the square of the lapse function in each point of view. Since null geodesics are invariant under spacetime conformal transformations, one may re-express the spatial equation of motion for a massless test particle in terms of the new line element, thus absorbing the scalar potential part of the gravitoelectric force term into the spatial geometry itself (since the rescaled metric has a unit lapse). In the case of a static spacetime with a nonlinear reference frame adapted to the threading congruence of a timelike Killing vector field, so that the shift field is zero and the adapted coordinates are Gaussian normal with respect to the new geometry, the spacetime geodesics of the rescaled spacetime metric project down to geodesics of the observer quotient space with the optical metric. Thus the geodesics of the optical spatial metric on this space are the paths of light rays, and the optical geometry measures deviations from these paths. Of course timelike geodesics are not invariant under conformal transformations, so the equations of motion of a massive test particle still contain an explicit gravitoelectric term when re-expressed in this way.

This idea, closely related to the general relativistic Fermat principle[37] and older discussions of Møller,[38] is the origin of a long series of papers by Abramowicz and coworkers. The key difficulty in understanding their calculations associated with this idea lies in the obscure representation both of the intrinsic derivative along the test particle world line and of the kinematical quantities of the observer congruence (which are not clearly identified) in terms of nongeometrical derivatives of a vector field on spacetime possessing the given world line as an integral curve. The very special circumstances of circular orbits in their applications also give false impressions of the general case; in particular the various centripetal accelerations and gravitoelectric and gravitomagnetic forces there are all transverse to the direction of relative motion, and so reside in the intersection of the local rest spaces of the observers and the test world line. The most recent formulation of this work[14] is simply the hypersurface point of view description of the decomposition of the spatial projection of the 4-acceleration of the test world line.

The details of the spacetime conformal transformation are straightforward and are discussed in the appendix. The new spatial equation of motion in the threading point of view for a massless test particle following a null geodesic turns out to be

$$\begin{aligned} &\tilde{D}_{(\mathrm{tem})}(P,m)\tilde{p}(P,m)_\alpha/d\tau_{(P,m)} \\ &= M^{-1}\tilde{E}(P,m)[-\pounds(m)_{e_0}M_\alpha + H_{(\mathrm{tem}\flat)}(m)_{\alpha\beta}\nu(P,m)^\beta] \ , \end{aligned} \tag{15.1}$$

where the flat notation $\flat$ in the gravitomagnetic field is only needed for the Lie case $_{\mathrm{tem=lie}}$, while in the slicing point of view it is

$$\tilde{D}_{(\mathrm{lie})}(P,n,e_0)\tilde{p}(P,n)_\alpha/d\tau_{(P,n)} = N^{-1}\tilde{E}(P,n)H_{(\mathrm{lie}\flat)}(n,e_0)_{\alpha\beta}\nu(P,n)^\beta \ . \tag{15.2}$$

The covariant rather than contravariant form of this equation is given for comparison with the form often found in the literature.[14] Here the explicit factors of the lapse correct for the proper time and the other untransformed spatial quantities still present.

Thus in the case of a static spacetime with an orthogonal slicing and threading adapted to the timelike vorticity-free Killing vector field, the threading and slicing points of view coincide ($u = m = n$), the shift and expansion tensor both vanish, all the various total spatial covariant derivatives agree, and the equation of motion reduces to the geodesic equation in the time-independent geometry of the observer quotient space

$$\tilde{D}_{(\mathrm{tem})}(P,u)\tilde{p}(P,u)^\alpha/d\tau_{(P,u)} = 0 \ . \tag{15.3}$$

In this special case only, one can interpret the optical total spatial covariant derivative as measuring the deviation of particle motion from "optically straight line paths" in the quotient space, as advocated by Abramowicz et al.[3–5] In the stationary case additional gravitomagnetic tensor effects deflect the null geodesics from the optical spatial geodesics in the observer quotient spaces.

The spatial equation of motion for massive test particles may also be expressed in terms of the conformally rescaled quantities. Using the results of the appendix one finds for the threading point of view

$$\begin{aligned} &\tilde{D}_{(\mathrm{tem})}(U,m)\tilde{p}(U,m)_\alpha/d\tau_{(U,m)} \\ &\quad = -\gamma(U,m)^{-1}\nabla(m)_\alpha \ln M - \gamma(U,m)\pounds(m)_{e_0}M_\alpha \\ &\qquad + \gamma(U,m)H_{(\mathrm{tem}\flat)}(m)_{\alpha\beta}\nu(U,m)^\beta + F(U,m)_\alpha \ , \end{aligned} \tag{15.4}$$

where the flat subscript is only needed in the Lie case. Correspondingly the expression for the spatial projection of the test particle 4-acceleration (just the gamma factor times the apparent three-acceleration) can be written

$$\begin{aligned} \gamma(U,m)A(U,m)_\alpha \quad = \quad & \gamma(U,m)\hat{\nu}(U,m)_\alpha dp(U,m)/d\tau_{(U,m)} \\ & +\gamma(U,m)^2[\nu(U,m)^2 D_{(\mathrm{tem})}(U,m)\hat{\nu}(U,m)_\alpha/d\ell_{(U,m)} \\ & +\nabla(m)_\alpha \ln M + \pounds(m)_{e_0}M_\alpha \\ & -H_{(\mathrm{tem}\flat)}(m)_{\alpha\beta}\nu(U,m)^\beta] \ . \end{aligned} \tag{15.5}$$

This in turn can be rewritten in terms of the optical metric and the natural conformally rescaled quantities introduced in the appendix as

$$\begin{aligned}\gamma(U,m)A(U,m)_\alpha \quad &= \quad \gamma(U,m)\tilde{\tilde{\nu}}(U,m)_\alpha d\tilde{p}(U,m)/d\tau_{(U,m)} \\ &\quad +\gamma(U,m)^2\{\nu(U,m)^2\tilde{D}_{(\mathrm{tem})}(U,m)\tilde{\tilde{\nu}}(U,m)_\alpha/d\tilde{\ell}_{(U,m)} \\ &\quad +\gamma(U,m)^{-2}\nabla(m)_\alpha \ln M + \pounds(m)_{e_0} M_\alpha \\ &\quad -H_{(\mathrm{tem}\flat)}(m)_{\alpha\beta}\nu(U,m)^\beta\} \ . \end{aligned} \tag{15.6}$$

Similarly, expressing the spatial projection of the 4-acceleration in the hypersurface point of view leads to

$$\begin{aligned}\gamma(U,n)A(U,n)^\alpha \quad &= \quad \gamma(U,n)\tilde{\tilde{\nu}}(U,n)^\alpha d\tilde{p}(U,n)/d\tau_{(U,n)} \\ &\quad +\gamma(U,n)^2[\nu(U,n)^2\tilde{D}_{(\mathrm{lie})}(U,n)\tilde{\tilde{\nu}}(U,n)^\alpha/d\tilde{\ell}_{(U,n)} \\ &\quad +\gamma(U,n)^{-2}\nabla(n)^\alpha \ln N \\ &\quad -H_{(\mathrm{lie})}(n)^\alpha{}_\beta\nu(U,n)^\beta] \\ &= \quad \gamma(U,n)\tilde{\tilde{\nu}}(U,n)^\alpha d\tilde{p}(U,n)/d\tau_{(U,n)} \\ &\quad +\gamma(U,n)^2[\nu(U,n)^2\tilde{D}_{(\mathrm{lie}\flat)}(U,n)\tilde{\tilde{\nu}}(U,n)^\alpha/d\tilde{\ell}_{(U,n)} \\ &\quad +\gamma(U,n)^{-2}\nabla(n)^\alpha \ln N] \end{aligned} \tag{15.7}$$

and in the slicing point of view

$$\begin{aligned}\gamma(U,n)A(U,n)_\alpha \quad &= \quad \gamma(U,n)\tilde{\tilde{\nu}}(U,n)^\alpha d\tilde{p}(U,n)/d\tau_{(U,n)} \\ &\quad +\gamma(U,n)^2[\nu(U,n)^2\tilde{D}_{(\mathrm{lie})}(U,n,e_0)\tilde{\tilde{\nu}}(U,n)^\alpha/d\tilde{\ell}_{(U,n)} \\ &\quad +\gamma(U,n)^{-2}\nabla(n)^\alpha \ln N \\ &\quad -H_{(\mathrm{lie}\flat)}(n,e_0)^\alpha{}_\beta\nu(U,n)^\beta] \ . \end{aligned} \tag{15.8}$$

In each point of view the key difference between the original and the conformally rescaled versions of these equations is the removal of the gamma squared factor which multiplies the lapse derivative term, which comes about from the difference term between the two covariant derivatives and the identity (7.2). For purely transverse relative accelerated motion in which the spatial acceleration lies in the common rest subspace $LRS_U \cap LRS_u$ (as in circular motion), then gamma times the spatial acceleration is the 4-acceleration itself, which can therefore be expressed as the sum of the logarithmic gradient of the lapse plus gamma squared times an "optical centripetal acceleration"

$$\tilde{a}^{(\perp)}_{(\mathrm{tem})}(U,u)_\alpha = \tilde{\nu}(U,u)^2\tilde{D}_{(\mathrm{tem})}(U,u)\tilde{\tilde{\nu}}(U,u)_\alpha/d\tilde{\ell}_{(U,u)} \tag{15.9}$$

plus additional terms due to the gravitomagnetic vector force and possible temporal derivatives. This is in some sense the decomposition of Abramowicz, which focuses on the optical centripetal acceleration, used to define an "optical centrifugal force" as gamma squared times the sign reversal of the optical centripetal acceleration[5]

$$\tilde{f}^{(\perp)}_{(\mathrm{tem})}(U,u)_\alpha = -\gamma(U,u)^2\tilde{a}^{(\perp)}_{(\mathrm{tem})}(U,u)_\alpha \ . \tag{15.10}$$

The square of the gamma factor enters from the change of proper time between the observer and test particle in the second derivative defining the acceleration.

One can also introduce the optical relative curvatures $\tilde{\kappa}_{(\mathrm{tem})}(U,u)$ and $\tilde{\kappa}_{(\mathrm{lie})}(U,n,e_0)$ as the magnitude of the optical derivatives $\tilde{D}_{(\mathrm{tem})}(U,u)\tilde{\hat{\nu}}(U,u)^\alpha/d\tilde{\ell}_{(U,u)}$ and $\tilde{D}_{(\mathrm{lie})}(U,n,e_0)\tilde{\hat{\nu}}(U,n)^\alpha/d\tilde{\ell}_{(U,n)}$ respectively, but all of these will only be relevant in the static case. This leads to the "optically straight" world lines which play a key role in the "reversal of the centrifugal force" discussion of Abramowicz et al.

Note finally that presence of the inverse square of the gamma factor multiplying the logarithmic gradient of the lapse in the conformally rescaled force equations above shows clearly that the ultrarelativistic limit for geodesic motion approaches the free photon behavior in stationary spacetimes for which the Lie derivative term is zero. As the relative speed approaches 1 and the gamma factor becomes increasingly large, this term drops out and the observer arclength parametrization approaches the proper time parametrization. One is then left with a balance of the optical centripetal acceleration and the gravitomagnetic force as occurs for photons, as noted by Abramowicz et al[14] for the case of circular orbits in stationary axially symmetric spacetimes.

In a long review,[12] both the threading and hypersurface points of view were mentioned in the context of stationary axisymmetric spacetimes, and both points of view were applied to the Kerr spacetime in separate articles,[10, 11] although the ambiguity of the individual force terms in this approach was not addressed until the gauge-fixing discussion of a later version of the force decomposition.[13] Although it is not easy to decipher (complicated by sign inconsistencies and minor errors), the subsequent presentation of the Abramowicz et al definition of noninertial forces for general spacetimes[14] is just the hypersurface point of view decomposition (15.7) of the sign-reversed spatial projection of the test particle 4-acceleration expressed in terms of the Lie total spatial covariant derivative and the contravariant relative velocity, with the gravitoelectric force and longitudinal and transverse relative accelerations shuffled among themselves by the conformal transformation. They refer to the latter two as the Euler and centrifugal forces when sign-reversed and conformally shuffled. The gravitomagnetic tensor force is entirely due to the expansion tensor, which they refer to as the Coriolis force with its extra gamma factor, while their gravitational force is just the gravitoelectric vector field itself without the gamma factor which occurs in the gravitoelectric force. The "ACL gauge" they have chosen for the extension of U^α off its worldline in the hypersurface point of view in order to make their derivative expressions meaningful is just $\nabla_{(\mathrm{lie})}(n)\hat{\nu}(U,n)_\alpha = 0$ (inconsistent with their covariant derivative formula for this derivative). One could also repeat their discussion with the equally valid but distinct corresponding contravariant gauge condition $\nabla_{(\mathrm{lie})}(n)\hat{\nu}(U,n)^\alpha = 0$, as well as switch to the threading point of view as alluded to in an earlier article.[12]

None of their forces coincide with the relative forces measured by the observers, and in the case of more general motion in which the various forces (apart from the longitudinal relative acceleration) are no longer all transverse, they do not lie in the local rest space of the test world line, so their interpretation in terms of forces that would be seen with respect to axes which comove with the test world line (apart from proper time adjustments) are no longer valid. This inconsistency appears in the claim that the various forces they introduce lie in the "comoving frame of the particle"[13] when in fact they lie in the local rest space of the observer (the exception being the intersection of the two subspaces orthogonal to the relative 2-plane of the motion).

Perhaps one can best characterize the limitations of the approach and its development by saying that it was born in a very special context and then attempts were made to generalize it, rather than realizing it as a specialization of an already general approach for arbitrary spacetimes, the tools of which have been around for a long time, but simply lacked a unifying umbrella. In particular, the key acceleration potential equation $u^\beta \nabla_\beta u_\alpha = \pm \nabla_\beta \Phi$ underlying this approach is inappropriate for nonstationary spacetimes where an additional spatial projection is needed for the gradient term and an additional Lie derivative of a vector potential is needed for a threading point of view. However, this said, the original application to static spacetimes where the optical centripetal acceleration does reverse when the relative signed optical curvature expression $\tilde{\kappa}(\phi, u)^{\hat{\rho}}$ associated with the ϕ coordinate circles changes sign is a very beautiful geometrization of the relative motion of massive and massless test particles, for which credit is clearly due for its recognition and description.

16. Concluding Remarks

By implementing relatively straightforward ideas about special relativistic space-plus-time splitting in the context of general relativity, using a notation which allows one to examine all the possibilities for generalizing concepts which do not have unambiguous extensions into the more general arena, a foundation has been built which enables one to analyse any particular problem that involves the description of idealized observations in a given spacetime. Not only are such questions fascinating, but their answers are often sufficiently subtle that much confusion has arisen even in the case of relatively simple spacetimes. Indeed the case of "rigid rotation" in flat spacetime itself still leads people astray in their attempts to come to terms with such questions.

Armed with the present tools, one can examine the traditional test cases for investigating these ideas,[39] namely Minkowski spacetime in rotating coordinates, the black hole spacetimes of Kerr and Schwarzschild, and the spacetime which was the first to dramatically challenge our intuition about rotation in general relativity nearly half a century ago, the Gödel spacetime. This will be done in a companion article, leading to a much clearer understanding not only of these spacetimes but of the tools themselves for studying other spacetimes.

Acknowledgements

We thank Remo Ruffini of the International Center for Relativistic Astrophysics at the University of Rome and Francis Everitt of the Gravity Probe B Relativity Mission group at Stanford University for their support and encouragement of this work, as well as the many relativists before us who have laid the groundwork for our analysis.

A. Conformal transformations

A.1. General considerations

Following the abstract index notation of Wald,[40] suppose one has a pair of conformally related metrics

$$\tilde{g}_{ab} = \sigma^2 g_{ab} \ , \tag{A.1}$$

each with its associated symmetric connection ∇ and $\tilde{\nabla}$. Then under the additional conformal transformation of a vector

$$\tilde{v}^a = \sigma^{-1} v^a \ , \quad \tilde{v}_a = \sigma v_a \ , \quad \tilde{v}^a \tilde{v}_a = v^a v_a \tag{A.2}$$

which preserves its magnitude, Eq. (D.5) of Wald

$$v^a \tilde{\nabla}_a v^b = v^a \nabla_a v^b + (2v^b v^c - g^{bc} v^d v_d) \nabla_c (\ln \sigma) \ , \tag{A.3}$$

is easily transformed to

$$\begin{aligned} \sigma^2\, \tilde{v}^a \tilde{\nabla}_a \tilde{v}^b &= v^a \nabla_a v^b + (v^b v^c - g^{bc} v^d v_d) \nabla_c (\ln \sigma) \\ &= v^a \nabla_a v^b - v^d v_d P(v)^{bc} \nabla_c (\ln \sigma) \ , \end{aligned} \tag{A.4}$$

where the last equality is only valid in the case $v^d v_d \neq 0$. $P(v)^a{}_b$ is just the tensor which projects orthogonal to v^a in that case, necessary since the either covariant derivative of a unit vector must be orthogonal to its direction, given that the metric is covariant constant. Using the latter fact, one obtains the covariant form of this equation

$$\begin{aligned} \tilde{v}^a \tilde{\nabla}_a \tilde{v}_b &= v^a \nabla_a v_b + (v_b v^a - \delta^a{}_b v^c v_c) \nabla_a (\ln \sigma) \\ &= v^a \nabla_a v_b - v^c v_c P(v)^a{}_b \nabla_a (\ln \sigma) \ . \end{aligned} \tag{A.5}$$

A.2. Conformal transformations of spatial quantities

For a given family of test observers with 4-velocity u^α and a given timelike world line of a test particle with 4-velocity U^α, consider a conformal transformation of the spatial metric

$$\tilde{P}(u)_{\alpha\beta} = \sigma^2 P(u)_{\alpha\beta} \ . \tag{A.6}$$

One can introduce a new spatial covariant derivative associated with the new spatial metric and use it to re-express the various total covariant derivatives of the relative velocity of a test particle. For our present purposes this is really only useful in the context of a nonlinear reference frame where the spacetime metric may be conformally rescaled by the inverse square of the lapse function, corresponding to a conformal factor σ equal to the reciprocal of the lapse function, $\sigma = M^{-1}$ or $\sigma = N^{-1}$ respectively (the optical gauge), leading to the optical spatial metric whose components in an observer-adapted frame have been denoted by γ_{ab} and g_{ab} respectively. However, the "antioptical" gauge $\sigma = M$ or $\sigma = N$ respectively is important in certain analyses of the gravitational field equations themselves in the case of stationary spacetimes and in post-Newtonian approximations.

From their definitions, the spatial arclength parameter, the relative speed, and the relative velocity unit vector must transform in the following way

$$\begin{aligned} &d\tilde{\ell}_{(U,u)}/d\ell_{(U,u)} = \sigma \ , \\ &\tilde{\nu}(U,u) = d\tilde{\ell}(U,u)/d\tau_{(U,u)} = \sigma d\ell(U,u)/d\tau_{(U,u)} = \sigma\nu(U,u) \ , \\ &\tilde{\hat{\nu}}(U,u)^\alpha = \sigma^{-1}\hat{\nu}(U,u)^\alpha \ , \end{aligned} \tag{A.7}$$

implying that the relative velocity vector

$$\tilde{\nu}(U,u)^\alpha = \nu(U,u)^\alpha = P(u)^\alpha{}_\beta dx^\beta/d\tau_{(U,u)} \tag{A.8}$$

is invariant.

One may also introduce the conformally rescaled energy and momentum (per unit mass)

$$\begin{aligned} \tilde{E}(U,u) &= \sigma^{-1}E(U,u) = \sigma^{-1}\gamma(U,u) \ , \\ \tilde{p}(U,u) &= \sigma^{-1}p(U,u) = \sigma^{-2}\gamma(U,u)\tilde{\nu}(U,u) \ , \\ \tilde{p}(U,u)^\alpha &= \sigma^{-2}p(U,u)^\alpha \\ \tilde{p}(U,u)_\alpha &= p(U,u)_\alpha \end{aligned} \tag{A.9}$$

satisfying

$$\tilde{E}(U,u)^2 - \tilde{p}(U,u)^2 = \sigma^{-2} \ . \tag{A.10}$$

This choice for the conformal scaling of these last two quantities is made so that in the special case of a Killing observer where $u^\alpha = M^{-1}\xi^\alpha$, with σ^{-1} taken to be the magnitude M of the Killing vector ξ^α, then the conformally rescaled spatial metric is the optical spatial metric and the conformally rescaled energy is the conserved quantity $\tilde{E}(U,u) = -p(U)^\alpha\xi_\alpha$ which is constant along the test particle world line, while a Killing component of the covariant spatial momentum remains a conserved quantity if it is not rescaled.

Finally, for a null geodesic where the equations of motion are conformally invariant, the new affine parameter can be defined by $d\tilde{\lambda}_P/d\lambda_P = \sigma^2$ as in Eq. (D.6) of Wald, leading to this same choice of conformal transformation for the momentum 4-vector $P^\alpha = dx^\alpha/d\lambda_P$ and its corresponding 1-form as for the spatial momentum and its 1-form in the case of a massive particle. This is easily seen from the action which gives the equations for affinely parametrized null geodesics as its Lagrangian equations

$$\int g_{\alpha\beta}(dx^\alpha/d\lambda_P)(dx^\beta/d\lambda_P)\,d\lambda_P \ . \tag{A.11}$$

This choice of transformation for λ_P leaves the action invariant.

One may introduce a new spatial covariant derivative $\tilde\nabla(u)_\alpha$ associated with the conformally rescaled spatial metric by using the appropriate difference connection as in Eqs. (D.3) and the sign reversal of (3.1.7) of Wald[40]

$$\begin{aligned} \tilde\nabla(u)_\alpha X_\beta &= \nabla(u)_\alpha X_\beta - X_\gamma \mathcal{C}^\gamma{}_{\alpha\beta} \ , \\ \mathcal{C}^\gamma{}_{\alpha\beta} &= [2\delta^\gamma{}_{(\alpha}\nabla(u)_{\beta)} - g_{\alpha\beta}\nabla(u)^\gamma](\ln\sigma) \ . \end{aligned} \tag{A.12}$$

This in turn may be used to introduce a conformally rescaled total spatial covariant derivative of each type, by replacing the spatial covariant derivative in the definition valid for vector fields by the conformally rescaled derivative. For example, for a spatial covector one has

$$\tilde D_{(\rm tem)}(U,u)X_\beta/d\tau_{(U,u)} = D_{(\rm tem)}(U,u)X_\beta/d\tau_{(U,u)} - X_\gamma \mathcal{C}^\gamma{}_{\alpha\beta}\nu(U,u)^\alpha \ . \tag{A.13}$$

One can easily re-express the three-acceleration or rate of change of spatial momentum in terms of this new derivative, using an immediate consequence of Eq. (A.12)

$$X^\alpha\tilde\nabla(u)_\alpha X_\beta = X^\alpha\nabla(u)_\alpha X_\beta - X^\alpha X_\alpha \nabla(u)_\beta(\ln\sigma) \ . \tag{A.14}$$

For example, for a congruence of test particle world lines one would have

$$\begin{aligned} D_{(\rm tem)}(U,u)p(U,u)_\beta/d\tau_U &= E(U,u)\nabla_{(\rm tem)}(u)p(U,u)_\beta + p(U,u)^\alpha\nabla(u)_\alpha p(U,u)_\beta \\ &= \tilde D_{(\rm tem)}(U,u)\tilde p(U,u)_\beta/d\tau_U + p(U,u)^2\nabla(u)_\beta(\ln\sigma) \ , \end{aligned} \tag{A.15}$$

where

$$\begin{aligned} \tilde D_{(\rm tem)}(U,u)\tilde p(U,u)_\beta/d\tau_U &= E(U,u)\nabla_{(\rm tem)}(u)p(U,u)_\beta + p(U,u)^\alpha\tilde\nabla(u)_\alpha p(U,u)_\beta \\ &= D_{(\rm tem)}(U,u)p(U,u)_\beta/d\tau_U - p(U,u)^2\nabla(u)_\beta(\ln\sigma) \end{aligned} \tag{A.16}$$

defines the rescaled total spatial covariant derivatives for either a congruence of test particle world lines or a single such world line respectively. Raising the index on these equations introduces an extra term from the temporal derivative of the conformal factor. These formulas yield the results for the threading and hypersurface points of view in the optical gauge for $u = m, n$, while replacing n by n, e_0 in the appropriate places in the hypersurface Lie form of these equations yields the slicing version.

One can similarly transform the derivative of the unit velocity vector needed to evaluate the relative centripetal acceleration, leading to the conformally rescaled quantities corresponding to the relative curvature and radius of curvature and the relative centripetal acceleration. If $\hat{\nu}(U,u)^\alpha$ were actually a vector field on spacetime rather than being defined only along a single world line, one could decompose its spatially projected intrinsic derivative in the following way

$$\begin{aligned} &D_{(\text{tem})}(U,u)\hat{\nu}(U,u)_\beta/d\ell_{(U,u)} \\ &\quad = [1/\nu(U,u)]D_{(\text{tem})}(U,u)\hat{\nu}(U,u)_\beta/d\tau_{(U,u)} \\ &\quad = [1/\nu(U,u)][\nabla_{(\text{tem})}(u) + \nu(U,u)\hat{\nu}(U,u)^\alpha\nabla(u)_\alpha]\hat{\nu}(U,u)_\beta \ . \end{aligned} \tag{A.17}$$

Now re-express this in terms of the conformally rescaled quantities using the results of the first section of the appendix for the unit velocity vector to re-express the spatial covariant derivative in it in terms of a conformally rescaled derivative $\tilde{\nabla}(u)_\alpha$. One finds

$$\begin{aligned} &D_{(\text{tem})}(U,u)\hat{\nu}(U,u)_\beta/d\ell_{(U,u)} \\ &\quad = \tilde{D}_{(\text{tem})}(U,u)\tilde{\hat{\nu}}(U,u)_\beta/d\tilde{\ell}_{(U,u)} \\ &\quad\ + P_u(U,u)^{(\perp)}{}_\beta{}^\alpha\nabla(u)_\alpha \ln\sigma - \hat{\nu}(U,u)_\beta/\nu(U,u)\,\nabla_{(\text{fw})}(u)\ln\sigma \ , \end{aligned} \tag{A.18}$$

where the first of the following equalities

$$\begin{aligned} &\tilde{D}_{(\text{tem})}(U,u)\tilde{\hat{\nu}}(U,u)_\beta/d\tilde{\ell}_{(U,u)} \\ &\quad = [1/\tilde{\nu}(U,u)][\nabla_{(\text{tem})}(u) + \tilde{\nu}(U,u)\tilde{\hat{\nu}}(U,u)^\alpha\tilde{\nabla}(u)_\alpha]\tilde{\hat{\nu}}(U,u)_\beta \\ &\quad = D_{(\text{tem})}(U,u)\hat{\nu}(U,u)_\beta/d\ell_{(U,u)} \\ &\quad\ - P_u(U,u)^{(\perp)\alpha}{}_\beta\nabla(u)_\alpha\ln\sigma + \hat{\nu}(U,u)_\beta/\nu(U,u)\,\nabla_{(\text{fw})}(u)\ln\sigma \end{aligned} \tag{A.19}$$

defines the equivalent action of the new derivative on a congruence of test particle world lines, while the second defines it for a single such world line.

The magnitude of this conformal derivative of the conformal unit velocity defines the conformally rescaled relative curvature and its reciprocal the conformally rescaled radius of curvature, for the ordinary and corotating Fermi-Walker cases. Multiplying the last relationship by the conformal square of the velocity gives the conformally rescaled relative centripetal acceleration and its relationship to the original one as long as the total spatial covariant derivative is orthogonal to the relative direction of motion

$$\begin{aligned} \tilde{a}^{(\perp)}_{(\text{tem})}(U,u)_\beta \ &= \ \sigma^2[a^{(\perp)}_{(\text{tem})}(U,u)_\beta - \nu(U,u)^2 P_u(U,u)^{(\perp)\alpha}{}_\beta\nabla(u)_\alpha\ln\sigma \\ &\quad + \nu(U,u)_\beta\nabla_{(\text{fw})}(u)\ln\sigma] \ , \quad {\scriptstyle \text{tem}=\text{fw},\,\text{cfw}} \ . \end{aligned} \tag{A.20}$$

Note that unless the observer temporal derivative of the conformal factor is zero, the "conformal derivative" of the unit velocity is not orthogonal to the unit velocity itself, even if it is before the conformal rescaling. The same is true of the conformally rescaled relative centripetal acceleration.

In the stationary case with σ chosen for the optical gauge, this temporal derivative vanishes while the spatial derivative produces the gravitoelectric field, and this relationship can be rewritten in the form (using first Eq. (7.2) and then Eqs. (9.1)–(9.4))

$$\begin{aligned}
&\gamma(U,u)^2\sigma^{-2}\tilde{a}^{(\perp)}_{(\mathrm{tem})}(U,u)_\beta \\
&= P_u(U,u)^{(\perp)\,\alpha}{}_\beta\{\gamma(U,u)^2[a^{(\perp)}_{(\mathrm{tem})}(U,u)_\alpha - g(u)_\alpha] + g(u)_\alpha\} \qquad (A.21)\\
&= P_u(U,u)^{(\perp)\alpha}{}_\beta[a(U)_\alpha + g(u)_\alpha + \gamma(U,u)^2 H_{(\mathrm{tem})}(u)_{\alpha\delta}\nu(U,u)^\delta]\ .
\end{aligned}$$

Thus if in addition the transverse gravitomagnetic force is zero, one sees that the optical relative centripetal acceleration changes sign when the transverse spatial projection of the test particle acceleration just balances the transverse gravitoelectric field, i.e., exactly opposes the transverse observer acceleration

$$\begin{aligned}
\gamma(U,u)^2\sigma^{-2}\tilde{a}^{(\perp)}_{(\mathrm{tem})}(U,u)_\beta &= P_u(U,u)^{(\perp)\alpha}{}_\beta[a(U)_\alpha - a(u)_\alpha]\ , \qquad (A.22)\\
\gamma(U,u)^2 a^{(\perp)}_{(\mathrm{tem})}(U,u)_\beta &= P_u(U,u)^{(\perp)\alpha}{}_\beta[a(U)_\alpha - \gamma(U,u)^2 a(u)_\alpha]\ ,
\end{aligned}$$

where for comparison the analogous relation for the relative centripetal acceleration itself is given (just the transverse analog of the static case longitudinal acceleration relation (3.13) of[30]). For static circular orbits, the optical relative centripetal acceleration then becomes outward pointing when the transverse test particle 4-acceleration becomes larger in magnitude than the transverse test observer 4-acceleration. This is the famous effect of the "reversal of the centrifugal force" motivating the work of Abramowicz et al. The additional squared gamma factor in the relative centripetal acceleration compared to the optical relative centripetal acceleration prevents the reversal in the former case.

References

1. R.T. Jantzen, P. Carini and D. Bini, *Ann. Phys. (N.Y.)* **215**, 1 (1992).
2. N.V. Mitskievich, "Relativistic Physics in Arbitrary Reference Frames," electronic preprint gr-qc/9606051, 1996.
3. M.A. Abramowicz, B. Carter, and J.P. Lasota, *Gen. Relativ. Grav.* **20**, 1173 (1988).
4. M.A. Abramowicz, and A.R. Prasanna, *M.N.R.A.S.* **245**, 720 (1990).
5. M.A. Abramowicz, *M.N.R.A.S.* **245**, 733 (1990).
6. M.A. Abramowicz, in *1989 Summer School in High Energy Physics and Cosmology*, J.C. Pati, S. Randjbar-Daemi, E. Sezgin, Q. Shafi (editors), World Scientific, Singapore, 1990.
7. M.A. Abramowicz, and J. Bicak, *Gen. Relativ. Grav.* **23**, 941 (1991).
8. M.A. Abramowicz, *M.N.R.A.S.* **256**, 710 (1992).
9. M.A. Abramowicz, J.C. Miller, and Z. Stuchlik, *Phys. Rev.* **D47**, 1440 (1993).
10. A.P. Prasanna, and S.K. Chakrabarti, *Gen. Relativ. Grav.* **22**, 987 (1990).
11. S. Iyer and A.P. Prasanna, *Class. Quantum Grav.* **10**, L13 (1993).
12. M.A. Abramowicz, in *The Renaissance of General Relativity*, G.F.R. Ellis, A. Lanza, and J.C. Miller (editors), Cambridge University Press, Cambridge, 1993.
13. M.A. Abramowicz, P. Nurowski, N. Wex, *Class. and Quantum Grav.* **10**, L183 (1993).
14. M.A. Abramowicz, P. Nurowski, N. Wex, *Class. Quantum Grav.* **12**, 1467 (1995).
15. O. Semerák, *Nuovo Cim.* **110B**, 973 (1995).
16. R. Nayak and C.V. Vishveshwara, *Class. Quantum Grav.* **13**, 1783 (1996).

17. A.P. Prasanna, *Class. Quantum Grav.* **14**, 227 (1997).
18. F. de Felice, *M.N.R.A.S.* **252**, 197 (1991).
19. F. de Felice, and S. Usseglio-Tomasset, *Class. Quantum Grav.* **8**, 1871 (1991).
20. F. de Felice, and S. Usseglio-Tomasset, *Class. Quantum Grav.* **10**, 353 (1993).
21. F. de Felice, *Class. Quantum Grav.* **11**, 1283 (1994).
22. F. de Felice, *Class. Quantum Grav.* **12**, 1119 (1995).
23. C. Barrabès, B. Boisseau, and W. Israel, *Mon. Not. R. Astron. Soc.* **276**, 432 (1995).
24. O. Semerák, *Gen. Relativ. Grav.* **28**, 1151 (1996).
25. E. Honig, E.L. Schücking, and C.V. Vishveshwara, *J. Math. Phys.* **15**, 744 (1974).
26. R.D. Greene, E.L. Schücking, and C.V. Vishveshwara, *J. Math. Phys.* **16**, 153 (1975).
27. B.R. Iyer, and C.V. Vishveshwara, *Class. Quantum Grav.* **5**, 961 (1988).
28. B.R. Iyer, and C.V. Vishveshwara, *Phys. Rev.* **D48**, 5706 (1993).
29. R.K. Sachs and H. Wu, *General Relativity for Mathematicians*, Springer-Verlag, New York, 1977.
30. D. Bini, P. Carini, and R.T. Jantzen, *Class. Quantum Grav.* **12**, 2549 (1995).
31. W. Rindler and L. Mishra, *Phys. Lett.* **173A**, 105 (1993).
32. L. Mishra, *Class. Quantum Grav.* **11**, L97 (1994).
33. V.I. Arnold, *Classical Mechanics*, Second Edition, Springer-Verlag, New York, 1989.
34. R. Arnowit, S. Deser, and C.W. Misner, in *Gravitation: An Introduction to Current Research*, L. Witten (editor), Wiley, New York, 1962.
35. C.W. Misner, J.A. Wheeler and K.S. Thorne, *Gravitation*, W.H. Freeman, New York, 1973.
36. K.S. Thorne, R.H. Price, and D.A. Macdonald, *Black Holes: The Membrane Paradigm*, Yale University Press, New Haven, 1986.
37. V. Perlick, *Class. Quantum Grav.* **7**, 1319 and 1849 (1990).
38. C. Møller, *The Theory of Relativity*, First Edition, Oxford University Press, Oxford, 1952; Second Edition, 1972.
39. W. Rindler and V. Perlick, *Gen. Relativ. Grav.* **22**, 1067 (1990); *errata: Gen. Relativ. Grav.* **23**, 119 (1991).
40. R.M. Wald, *General Relativity*, University of Chicago Press, Chicago, 1984.

Corrections

This reformatted version contains two reference publication updates and one typo correction: line 4 after 15.10 (top of p. 31 in the original article) where "the optical derivatives $\tilde{D}\ldots$" should have the two uppercase D derivative symbols with an over tilde.